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ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. The articles in this volume deal with the instruction in geometry and arithmetic of mentally retarded pupils in the Soviet Union. These pupils attend special schools, called auxiliary schools, where they are trained in content that can later be related to specific job skills. Authors of the articles have attempted to identify the specific knowledge that the pupils possess and to design more effective instructional methods for increasing that knowledge. (Author/MK)

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SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS

VOLUME X

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THE UNIVERSITY OF CHICAGO

SE 030 457

SOVIET STUDIES
IN THE
PSYCHOLOGY OF LEARNING
AND TEACHING MATHEMATICS

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VOLUME X

TEACHING MATHEMATICS TO MENTALLY RETARDED CHILDREN

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PREFACE

The series Soviet Studies in the Psychology of Learning and Teaching Mathematics is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

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*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See Contemporary Soviet Psychology by Josef Brozek (Chapter 7 of Present-Day Russian Psychology, Pergamon Press, 1966).

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections *Izvestiya Akademii Pedagogicheskikh Nauk RSFSR* [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly *Sovetskaya Pedagogika* [Soviet Pedagogy], and the bimonthly *Voprosy Psikhologii* [Questions of Psychology]. Since 1963, the Academy has been issuing collection entitled *Novye Issledovaniya v Pedagogicheskikh Naukakh* [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.[*] The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking—stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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[*] See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses or seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were prepared jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

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INTRODUCTION

The articles in this volume deal with the instruction in geometry and arithmetic of mentally retarded pupils in the Soviet Union. These pupils attend special schools, called auxiliary schools, where they are trained in content that can later be related to specific job skills. Authors of the articles have attempted to identify the specific knowledge that the pupils possess and to design more effective instructional methods for increasing that knowledge.

In the first article Tishin cites visual geometry as being of much greater importance in the Soviet auxiliary schools than in the public schools. The purpose of Tishin's study was to determine the nature and extent of the auxiliary school pupil's geometric knowledge and to suggest methods for increasing this knowledge. The results of the investigation are reported for groups of six children from each of grades one through seven. An analogous investigation was carried out with normal children in lower grades in the public schools. In all, there were nineteen series of investigations: eleven on the knowledge of plane figures and eight on the knowledge of geometric bodies or solids.

Tishin carefully explains the methods he used and gives detailed results. He concludes, among other things that auxiliary school pupils can select geometric forms when given a model; that with slightly more difficulty, auxiliary school pupils can select geometric forms according to name, draw geometric figures, and model geometric bodies; and that public school pupils have less difficulty naming geometric forms than do auxiliary school pupils.

In Tishin's investigation of auxiliary school pupils' knowledge of square and cubic measure, he first asked students to compute the area of three rectangles and two squares. These figures were presented in order of increasing difficulty. The figures were drawn, and the pupils were required to measure to get the dimensions. The pupils could do many of the measurements and computations correctly; however, they were not always correct in indicating the units used in the computations or obtained in the answer.

In another series of investigations, Tishin gave the pupils word problems calling for the computation of square and rectangular measurements. The pupils were much more successful in obtaining correct answers to these problems, notwithstanding their verbal content. Here again, the public school pupils were deemed better at solving these problems than the auxiliary school pupils.

To determine their knowledge of cubic measure, the pupils were presented with three parallelepipeds and two squares and were told to determine the volume. They encountered many difficulties in determining the dimensions of the figures. However, when given word problems the pupils were able to solve them much more easily. Again, as in the investigation with square measure, the pupils were not consistent in their presentation of the units of the answer.

After exhaustively describing his investigations, Tishin gives concrete suggestions for improving the pupils' knowledge of geometry. He describes many exercises that can be used in any classroom where these concepts need to be taught. Tishin's article is long and detailed but can be easily read and understood by teachers. The ideas included in the sections on "Pedagogical Implications" can be adapted to one's own classroom.

Kuz'mina-Syromyatnikova, like Tishin, feels that the mentally retarded child needs a pedagogical treatment designed specifically for him and not merely adapted from that designed for a normal child. This idea is also held by many American researchers [1]. She builds a rather strong case for utilizing both visual and verbal means in teaching pupils to solve arithmetic problems.

In the investigation discussed in this article, the students were interviewed singly while solving problems and were questioned on the methods they used. Kuz'mina-Syromyatinikova concludes that pupils were much more successful in problem solving when they could recognize a link between the problem they were attempting to solve and some past experience. Many times, however, they wished to apply what they did in the past without any modification. She also found that in choosing the operations needed to get the correct solution, the pupils were influenced by the order of the numbers presented as well as by certain "guiding words," such as "he had in all" and "how many more?"

In discussing the use of visual aids in solving verbal problems, Kuz'mina-Syromyatnikova cautions teachers to choose aids carefully, depending on the specific problems being taught. She also gives a method of notation that can be used by pupils to aid them in their computation.

Kuz'mina-Syromyatnikova also investigated the relationship between the pupil's ability to reproduce the problem and his ability to solve the same problems. Unfortunately, half of the subjects were unsuccessful in reproducing the problem and none of them solved it. An experimental lesson presenting problems in verbal form and in verbal-visual form was taught. The approach utilizing both visual and verbal means was by far the most successful with auxiliary school pupils. When visual aids were coordinated with the verbal problems, the pupils were much more able to reproduce the problems and arrive at a solution than when the problems were presented only in verbal form.

In the final article in this volume, Khanutina reports a study designed to ascertain the knowledge of spatial features (big-small, long-short, etc.), the knowledge of quantitative relationships (greater-less, etc.), the ability to count to ten, and the ability to work with problems involving numbers from one to ten. Instructions were given verbally, and concrete materials were used whenever possible. Normal children could handle all tasks well, whereas mentally retarded students seemed to have difficulty with many of them.

Based on the preliminary investigation, methods for teaching the concepts involved in spatial and quantitative relationships, counting, and elementary problem solving were carefully designed and are explained in the concluding portion of the article.

This volume contains much that should interest teachers working with children who have learning difficulties. It has many concrete suggestions that can be adapted for the classroom.

REFERENCE

1. Connolly, Austin J. "Research in Mathematics Education and the Mentally Retarded," Arithmetic Teacher, Vol. 20, 1973, pp. 491-497.

INSTRUCTING AUXILIARY SCHOOL PUPILS IN VISUAL GEOMETRY[*]

P. G. Tishin

Mathematical knowledge is exceptionally important in a system for instructing pupils. Engels wrote: "Pure mathematics has as its object the spatial forms and quantitative relationships of the real world, and hence it is altogether real material [3:37]." Arithmetic is the study of quantitative relationships, and geometry is the study of spatial forms. In equipping pupils with mathematical knowledge, it would be incorrect to give the teaching of geometry a place secondary to that of arithmetic in the teaching process.

The history of the development of teaching visual geometry in the public elementary school[**] has shown that several methodologists have included elementary information from visual geometry in the general arithmetic course.

The volume of arithmetic material provided in the curriculum of the auxiliary school corresponds to the arithmetic material in the curriculum of the public elementary school. Hence, the elementary information of visual geometry in the arithmetic course in the auxiliary school should correspond to that in the arithmetic course in the public school.

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[*] From Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, 1952, Vol. 41, pp. 81-164. Translated by Michael Ackerman.

[**] In this article "public schools" are those open to the general public and are contrasted to "auxiliary schools," which admit only children who are retarded in some way (Ed.).

The pupils of the auxiliary school need more information from visual geometry than do the pupils of the public elementary school, since, after seven years of instruction, the former have received a definite amount of knowledge with which they set forth directly into life and into production. Moreover, an acquaintance with geometric forms expands the store of elementary ideas and concepts in mentally retarded children, in whom spatial relations are very weakly developed. The pupils of the auxiliary school need geometric knowledge in manual training classes, in classes in geography, science, drawing, and other subjects, but especially in professional training.

No form of work can proceed without using knowledge of elementary visual geometry. Knowledge of geometric material is widely used in sewing, in carpentry, in cardboard binding, and in several other types of professional training. Without measuring and drawing, without a knowledge of geometric forms and their relationships, a pupil is unable to carry out his work tasks.

Thus, mentally retarded children need a knowledge of visual geometry for the purpose of raising their general educational level, and also for an educational and practical goal. The program of the auxiliary school states that:

> In all grades--from the first to the seventh--it is necessary to conduct lessons whose goal is to provide the basic concepts of elementary geometry, to teach the pupils the simplest procedures of measuring and drawing [2:29-30].

In addition, the ability to use knowledge of visual geometry that one has obtained has enormous importance in the development of the mentally retarded child's personality.

Although the study of geometric material in the auxiliary school is stipulated by the educational curriculum, studies of visual geometry often have a formal character and do not fulfill the general requirements of teaching and educating pupils of the auxiliary school. This is because not much is known about the study of geometric material in the auxiliary school. The teachers of the auxiliary school, lacking special methodological direction, are forced to use the methodological literature on visual geometry in the public elementary school, which does not reflect the special characteristics of work with pupils of the auxiliary school.

An Investigation of Auxiliary School Pupils' Knowledge of Geometric Forms

1. The Problems and Methods of the Investigation

The goal of our investigation was to discover the basic nature of the process of studying geometric material in the auxiliary school.

The teaching of visual geometry to auxiliary school pupils pursues the following primary goals:

a) the development of spatial ideas and the expansion of the scope of ideas and concepts of geometric figures and bodies;
b) the mastery of a definite system of knowledge of visual geometry;
c) the development of abstract logical thinking;
d) the practical utilization of knowledge obtained in lessons in other school disciplines and in professional training;
e) the application of the knowledge, skills, and habits in practical work after completing school.

In this connection we posed the following problems:

a) to investigate the pupils' idea of geometric forms;
b) to discover the characteristics of the geometric knowledge of auxiliary school pupils; and
c) to determine ways for auxiliary school pupils to study geometric material.

In investigating geometric ideas we thought we should elucidate:

a) whether the pupils can draw geometric figures and model geometric bodies;
b) whether they know the names of the geometric forms;
c) whether they can find the geometric form corresponding to a given name; and

d) whether they can find the name corresponding to a given geometric form.

Our attempt to ascertain their knowledge involved the pupils' active participation. Not only did the pupils verbally define and name the geometric forms, but they supplemented their answers by drawing and tracing geometric figures, modeling geometric bodies, and tracing and cutting out figures.

The individual investigation was conducted with groups of six pupils from the first to the seventh grades. For a comparison, we conducted an analogous investigation in the lowest grades of a public school.[1] Each pupil's rate of progress was considered in selecting participants for the study.

Altogether we conducted eleven series of investigations in revealing the pupils' knowledge of geometric figures and eight series in clarifying their knowledge of geometric bodies. We followed a definite order closely, to reveal the pupils' real knowledge of geometric forms.[2]

2. Results of the Individual Investigation on Ascertaining the Auxiliary School Pupils' Knowledge of Geometric Forms

As a result of the investigation we obtained the following material, which we shall examine.

---

[1] The individual investigation was conducted in various schools and classes in order to clarify the special characteristics of the pupils in the auxiliary school and to avoid introducing the individual peculiarities of the instruction of particular teachers. To emphasize the material obtained we took only part of the data from the experiment; we selected the results of the investigation of groups of six pupils each from the first to the seventh grades. The pupils of other grades gave similar results. In our general conclusions we used the entire experimental material obtained.

[2] In our work we retained the terminology of the auxiliary school's program; that is, plane geometric figures were called simply geometric figures, and solid geometric figures were called geometric bodies. The pupils of the auxiliary school learn this terminology more easily.

The Pupils' Ability to Represent A Geometric Figure in a Drawing and to Model a Geometric Body

When the pupils were asked to sketch a circle, only one second grader did not carry out this task. The others drew a circle. The fourth graders used compasses, and the others drew freehand.

When asked to draw a square, some pupils made errors. One first grader drew a rectangle instead of a square, and another pupil drew a triangle. Four third graders drew triangles.

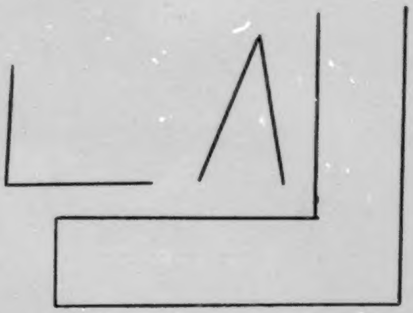

Figure 1

The children have a less accurate idea of a rectangle. Most pupils represented a rectangle incorrectly. Instead of a rectangle they drew a triangle, and three third graders drew a rectangle as a right angle or set square. (Figure 1).

The pupils' ideas of a triangle were correct in most cases, but some pupils drew a group of three triangles (Figure 2) instead of one, understanding by the term "triangle" a set square for mechanical drawing. One pupil drew three angles placed separately (Figure 3), and one pupil drew the representation shown in Figure 4.

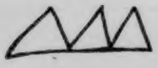

Figure 2

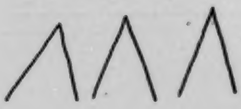

Figure 3

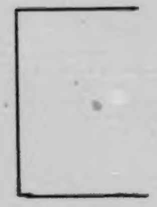

Figure 4

In drawing geometric figures the fourth graders used compasses and rulers, but in drawing a right triangle none of the pupils used a set square. In the first and second grades the best results were displayed in drawing a circle and a square, but the third graders gave better results in drawing a circle and a triangle. The pupils made more errors in drawing the related figures of square and rectangle. The pupils' idea of the circle was most accurate, then of the square, of the triangle, and of the rectangle.

In representing geometric figures in a drawing, the pupils in the highest grades of the auxiliary school (5-7) showed the following results. When asked to draw a circle, all the pupils of the highest grades did so, most of them using compasses. Almost none of the pupils (from different classes) drew freehand with a high degree of precision.

Mistakes were made in representing a square and a rectangle. Three fifth graders drew a rectangle instead of a square. All other pupils drew a square correctly, some freehand.

A significant portion of the pupils drew a rectangle correctly. The following mistakes were made in representing a rectangle: a fifth grader drew a triangle instead of a rectangle, one seven grader drew a square instead, and another a right triangle. In drawing a rectangle and a square, and in constructing a right angle, a set square was not used. The pupils were instructed: "Before you are pencils, ruler, set square, and compass; trace a circle, square, rectangle, and triangle." Then they were allotted complete freedom of action. Given this instruction and freedom, some drew figures with a ruler, and some freehand.

The pupils had a more accurate idea of the triangle than of the square or rectangle. All drew a triangle correctly. Thus in our investigation the pupils did better in representing the circle and the triangle than the square and the rectangle. The pupils of the auxiliary school confuse related figures more easily than contrasting figures.

In drawing geometric figures from a model the pupils made significantly fewer mistakes than in representing the same figures when their names were given. Almost all coped satisfactorily with the problem. All the pupils reproduced a triangle and circle precisely, and in the fourth grade they used ruler and compass in tracing these figures. Only one first grader drew a square instead of a rectangle, and only one second grader drew a rectangle instead of a square; the other pupils drew correctly.

In representing geometric figures the pupils in the higher grades made mistakes only in drawing a square and a rectangle, that is, in representing related figures. Most often the pupils confused the square and the rectangle, because there are no sharp differences between these figures as, for example, between the circle and the triangle.

Upon comparing the results of the investigation conducted in the auxiliary and public schools, we noted some differences and general tendencies: the pupils of the public school reproduced geometric figures more precisely. Almost all pupils of the normal school used compasses in drawing a circle, but in the auxiliary school they did not use compasses until the fourth grade. In neither school did any pupil use a set square to draw a square and rectangle. The pupils in both schools represented the circle and the triangle more accurately and the square and the rectangle less accurately.

The representations of geometric figures made by the pupils of the two schools differed in the precision and accuracy of lines. Figure 5 shows some of the figures made by a fourth grader in the public school.

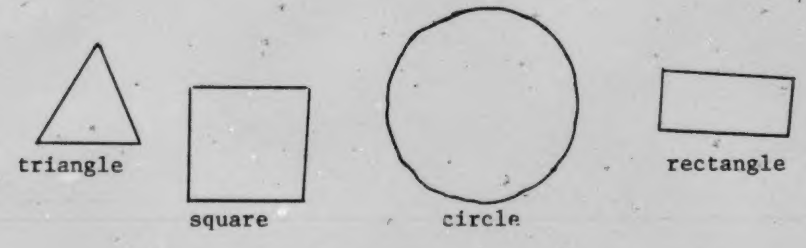

Figure 5

Figure 6 shows the same geometric figures drawn by a seventh grader in the auxiliary school.

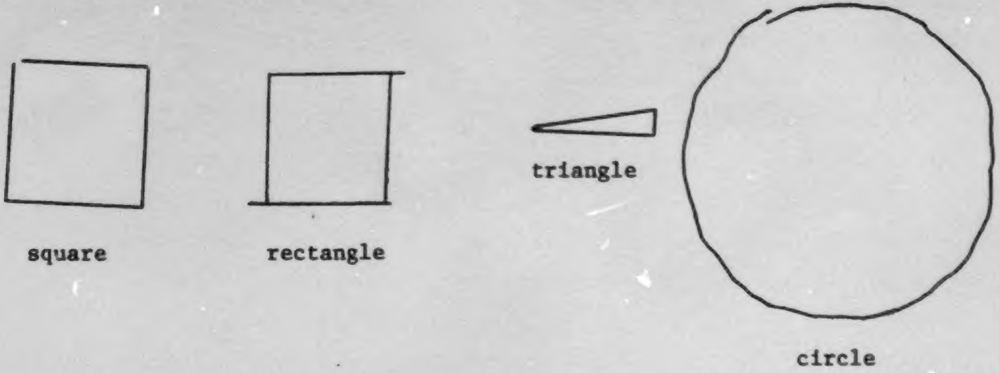

Figure 6

The pupils of the auxiliary school were also asked to make models of a series of geometric bodies. The results of their work were the following: out of 24 pupils of the lower grades (1-4) of the auxiliary school only four could model a cube. In one case a pupil gave up; the rest did it wrong by modelling a parallelepiped instead of a cube. In 14 cases the parallelepiped had a square base. In most "cubes" modelled, the edge of the square base was greater than the height. Some pupils modelled a parallelepiped with other bases. One modelled a cylinder.

The children had the most accurate idea of a sphere or ball. Only one third grader, R. A., modelled a cylinder instead; a second grader, G. V., modelled a disc.

Only half of the subjects could model a parallelepiped or square bar correctly. Three modelled it in the form of a whetstone, and the rest stated their inability to do this.

The children did not know the cylinder, cone, or pyramid, and thus in almost all cases gave up. Only third grader Ya. E. made a pyramid in the form of a toy, and third grader B. K. modelled a parallelepiped instead of a cylinder and pyramid, and a whetstone instead of a cone.

When the pupils of the upper grades of the auxiliary school were asked to model a cube, only half did it correctly. Three fifth graders and two sixth graders modelled a parallelepiped with a square base instead, and fifth grader P. N. could not model at all. Seventh grader R. M. modelled a parallelepiped.

All the pupils being examined modelled a ball, half correctly and half nearly so.

More than half modelled a parallelepiped. The others did not produce a model, since they had no idea of the given geometric body. Seventh grader R. M. modelled a long plate, and three fifth graders and four sixth graders did not model the given geometric body at all. The seventh graders no longer called a parallelepiped a beam, but a parallelepiped.

The pupils of the upper grades did not model a cylinder, a cone, or a pyramid. Some pupils tried to model a cone and a pyramid in the form of objects familiar to them. For example: two pupils of the fifth and seventh grades modelled a hammer instead of a cone, and a sixth grader modelled a truncated pyramid, but he also modelled a truncated cone instead of a pyramid. The other pupils did not try to model.

Almost no one modelled a pyramid. A sixth grader modelled a truncated cone instead, and one seventh grader modelled a toy pyramid, and another a horseshoe. The other pupils did not try to model, since they did not know the given geometric body. None of the pupils could model a cylinder, and no one even tried.

Making geometric bodies from a model caused significantly fewer difficulties, since the pupils could compare the models they were making with the given model.

Comparing the results obtained in the auxiliary and the public schools, we observed some general similarities, as well as qualitative differences. The pupils of both schools represented a ball more accurately than a cube, and a cube more accurately than a parallelepiped. A common characteristic of reproducing a cube is an insufficiently accurate idea of the given geometric body and failure to single out its distinctive features. The mistakes were homogeneous. With the aid of a model, the pupils reproduced not a cube but a parallelepiped with a square base.

The pupils of the public school represented geometric bodies more accurately than the pupils of the auxiliary school.

For comparison we give representations of the geometric bodies as modelled by the pupils of the auxiliary and the public schools. (Figures 7 and 8).

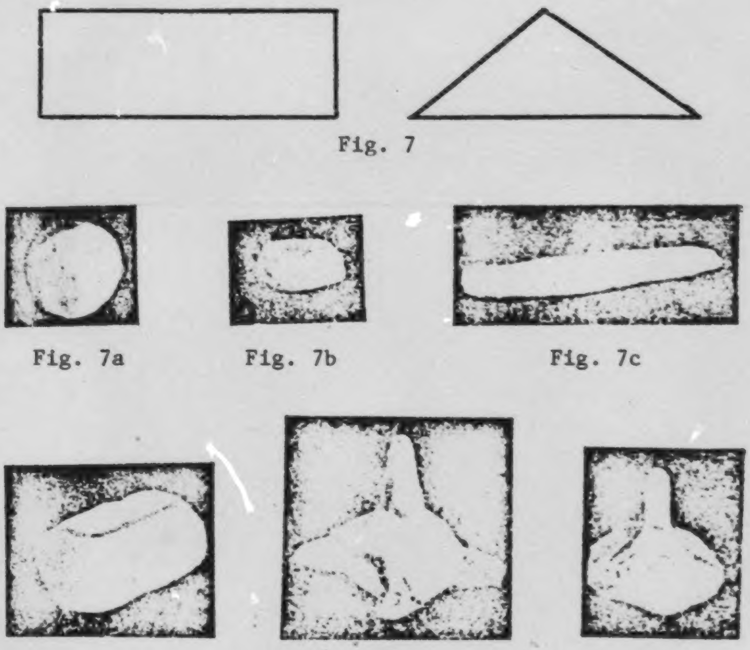

Fig. 7

Fig. 7a

Fig. 7b

Fig. 7c

Fig. 7d

Fig. 7e

Fig. 7f

Figure 7: Geometric Bodies Modelled by Pupils of the Auxiliary School

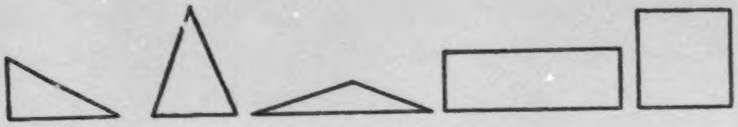

Fig. 8

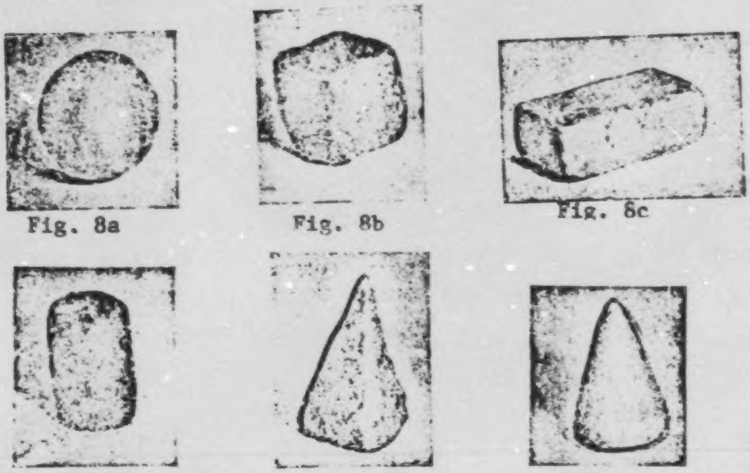

Fig. 8a Fig. 8b Fig. 8c

Fig. 8d Fig. 8e Fig. 9f

Figure 8: Geometric Bodies Modelled by Pupils of the Public School

Ascertaining the Pupils' Knowledge of the Names of Geometric Figures and Bodies

The pupils were shown geometric figures successively—circle, square, triangle and rectangle--and were asked to name the geometric figure presented. For the figures given, the pupils of the lower grades of the auxiliary school almost always gave more incorrect names.

The retarded children had a less mistaken idea of the circle than of the other figures. Most correct answers were: "little circle" (khruzhochek and kruzhok). The name kruzhochek was given by first through third graders, and kruzhok by fourth graders. The incorrect answers "little ball" or "round little ball" show that some children confuse circle and ball.

They also gave many correct answers for the triangle, calling it a "three-angle," according to the number of its elements. In determining this geometric figure, there is only one essential feature—the three angles. The fewer the essential features, the easier it is for the child to name the figure correctly if, in addition, there is a similarity between the name of the figure and its definition by essential features. Some first and second graders called the triangle a "mirror" or "a little house"; that is, they gave the geometric figure an objective interpretation. The third and fourth graders named a triangle in base position correctly. Two fourth graders called the triangle a "set square" or "angle-thing."

When a rectangle was presented, the first and second graders either did not name the given figure at all or gave it mistaken names, such as "little square," "little cube," "mirror," "little plant," "domino," "long thing," and so on. The third and fourth graders gave correct names. The children easily confused the rectangle with the square. Perceiving a square, the pupils called it a "four-angle" or "little four-angled thing." Thus, there was one essential feature at the basis of their definition--the four angles. Most mistakes consisted of the answer "little cube." Cube and square were confused because the face of a cube is square.

There were three essential features in the definition of the rectangle and square. The first feature was the four angles; the second, that all the angles were right angles; and the third--for the square--the equality of all sides, and for the rectangle the equality of opposite sides. But the pupils called the square and rectangle simply "four-angle"; that is, as the basis of their definition they chose only one essential feature.

In the Ul'yanovsk Auxiliary school we asked the children to name a rectangle and a triangle of oblong form (Figure 7).

The correct naming and recognizing of a square and a circle proceed differently than the naming and recognizing of a rectangle and of a triangle. This difference is because in the square and the circle only the scale of the figure can change, while in the rectangle and triangle, the relations of the sides can change. This influences, to some extent, the change of form of the figure (Figure 8).

When shown an oblong rectangle, the pupils answered "ruler," or "little ruler." There were significantly fewer correct answers than when the rectangle had been introduced, the latter having a form familiar to the children. Most pupils could not name an elongated triangle correctly although they had named the previous triangle correctly. The rectangle, because its form was like that of a square, was called a square.

In the naming of geometric figures we obtained the following data from the pupils of the upper grades:

When a circle was introduced, almost all the pupils named it correctly ("circle," "little circle," "round thing"), and only two seventh graders called a circle a "circumference."

When a square was introduced, most pupils named this correctly too. Three pupils (two fifth graders and one seventh grader) called a square a "four-angle" and two fifth graders called it a "little cube" and a "flat," that is, gave an incorrect name.

When a triangle was introduced most pupils called it a "three-angle." Two fifth graders called it an "angle-thing," and two slow seventh graders a "rectangle" and an "obtuse-angled thing."

When a rectangle was introduced the pupils gave more incorrect answers in comparison with the other figures. A sixth grader called a rectangle a "four-angle," and a fifth grader called it a "right-angled thing," and three seventh graders, a "square" and an "angle-thing."

The pupils of the upper grades already have a more accurate idea of circle, square, rectangle, and triangle than those of the lower grades. We asked the pupils to name three rectangles. One was 12 cm long, 2 cm wide; the second, 12.5 cm long and 7 cm wide; and the third, 9.5 cm long and 8.5 cm wide. Eight out of twelve pupils named the first figure correctly. One sixth grader called it a "four-angle," and one sixth grader said: "This is not a rectangle; it is like a rectangle; we didn't learn about such things." Two seventh graders called such a rectangle a "parallelepiped." All sixth graders named the second figure correctly, and three seventh graders also named it correctly. Besides this, one seventh grader called this rectangle a square; another, a parallelepiped; and a third said that he had forgotten. The third rectangle (close in form to a square) caused more difficulties for the pupils. Three sixth graders and two seven graders at first called this rectangle a square, but after measuring the sides they gave the correct names. Three sixth graders left it with the name "square"; and of the seventh graders, one called it a "four-angle"; another "a square," a third, "a cube"; and one, even after measuring the rectangle, called it a "parallelepiped." Thus, recognition depends on the relationship of the sides to a significant extent.

The naming of geometric bodies proceeded thus:

Most pupils of the lower grades of the auxiliary school named the ball correctly. Some pupils called it a "little circle" [kruzhok or kruzhochek], "little ball," or "nut"; that is, either they did not distinguish the geometric body from the figure sufficiently, or they named it after an object similar in form.

When a cube was introduced, most children called this body a "little cube," one pupil called it a "four-angled little cube," and only a few pupils called it a "square," or "little box," a "four-angled thing."

The children have a less accurate idea of the parallelepiped as a geometric body.

The pupils do not know the cone as a geometric body and call it a "little cap," a "hood," or the like, that is, they give an objective interpretation to the given geometric body.

The mentally retarded school children called the cylinder "a little jar," a "little box" and some called it a "circle" and a "little cube."

They called the pyramid "little house,", "side-cap," "tent," "tri-angle," "lighthouse," and the like.

The pupils of the upper grades of the auxiliary school gave the following answers:

When a "ball" was introduced, almost all the pupils named it correctly. The greatest percentage of correct answers came from sixth and seventh graders. Only one seventh grader,K. N., called a ball a "circle." One should also note that seventh graders I.Z. and M. R. had an inaccurate idea of a ball, which they called a "circle," "circumference," and "ball."

When a cube was presented the correct answers were: "cube,"--

mainly from seventh graders, and "little cube"—from fifth and sixth graders. Some fifth and sixth graders called a cube a rectangle and a square.

The pupils have less accurate knowledge of the parallelepiped. Most correct answers were given by seventh graders. One sixth grader called it a "beam," and a fifth grader called it a "little beam." For the most part, the fifth and sixth graders called a parallelepiped a rectangle, a square, or a matchbox.

In naming familiar geometric bodies the pupils of the upper grades made mistakes much less frequently. When a cone was introduced, one fifth grader named it correctly, and the remaining pupils either did not name it at all or gave mistaken names ("three-angle," "lighthouse," "circle," "cap," "tower"). Most names were names of objects similar in form ("lighthouse," "cap," "tower ") or names of geometric figures which are projections of the given geometric body ("triangle," "circle").

When a cylinder was introduced only one fifth grader and three seventh graders named it correctly, and the others either did not name it at all or gave it names of objects similar in form to a cylinder ("kind of a little barrel," "box," "round," "cup," "round jar," "tub"); one fifth grader called a cylinder a "circle," and seventh grader I. Z. called it a "perimeter." In their answers most pupils noted the distinction of the given geometric body, especially its round form.

In naming the pyramid the pupils followed the same pattern as in naming the cone; that is, only one sixth grader gave a correct answer, and the rest either gave no name or called it an object similar to the pyramid ("cap," "lighthouse"). Some pupils named it after geometric figures which are projections of the given geometric body ("triangle," "five-angle"), and some pupils called a pyramid a cone, basing this on some similarity between them.

Comparing the results of the investigation conducted in the public school, we noted the following:

A characteristic aspect, common to both the mentally deficient children and the pupils of the public school was the absence of a generalized word designating the given geometric form (figure or body).

The number of word names designating geometric forms is given in Table 1.

TABLE 1

NUMBER OF WORD NAMES USED TO DESIGNATE GEOMETRIC FORMS

Form	Number of Names Auxiliary School Pupils	 Public School Pupils
Circle	8	6
Square	10	8
Rectangle	16	7
Triangle	9	6
Sphere	6	5
Cube	5	4
Parallelepiped	12	11
Cone	12	12
Cylinder	15	10
Pyramid	12	12

The pupils of both schools have an insufficient idea of geometric forms. Table 2 shows the percent of correct names given to individual figures and bodies.

TABLE 2

PERCENT OF CORRECT NAMES GIVEN TO GEOMETRIC FORMS

Form	Auxiliary School Pupils	Public School Pupils
Plane		
Circle	40.8	83.3
Triangle	62.5	66.7
Square	20.8	50.0
Rectangle	16.7	20.8
Solid		
Sphere	72.2	87.5
Cube	83.3	87.5
Parallelepiped	25.0	37.5
Cylinder	0	12.5
Cone	0	4.2
Pyramid	0	8.3

Some pupils of the auxiliary school gave objective-geometric names to familiar forms, which the pupils of the public school did not. We observed an entirely different picture when we introduced unfamiliar geometric forms--for example, the cone, cylinder, and pyramid. The pupils of the public school also named these three geometric forms after objects.

Geometric figures are plane geometric forms. The pupils of the auxiliary school distinguish the plane form from the solid poorly, and they name the geometric figure after an object. Geometric bodies have three dimensions, and, by their form, recall an object; hence, the pupils of the public school named an unfamiliar geometric form of three dimensions after a similar object. They did not do this for the plane forms which they easily distinguished from the geometric bodies.

One should note that the pupils of the auxiliary school have a better idea of a solid geometric form than of plane forms. At the same time, they have unequal ideas of the solid forms. They have a better idea of the ball and cube, and a poorer one of the other solids. Of the plane forms, they have a better idea of circle and triangle, and a poorer one of square and rectangle.

Ascertaining the Abilities to Find the Corresponding Geometric Form from a Given Name.

The pupils were asked to find particular geometric forms among ones spread over a table, to repeat the name and to answer that this is the geometric form sought "or to tell all that they know about the geometric body." When the pupils of the lower grades of the auxiliary school were asked to find a circle, all pupils solved the problem correctly with the exception of one first grader, Kh. S.

The next most accessible form for the mentally deficient child is the triangle. Correct solutions were given by 22 out of 24 people, or 91.7%. The next form in degree of accessibility is the rectangle: 87.5% of the children's answers were correct. The most difficult figure for the pupils to perceive is the square. Only 66.7% gave the correct answers. For the most part they confused the square with the rectangle and the triangle.

In the selection of geometric bodies according to name from a collection which had been set out, we obtained the following results:

Twenty-four pupils chose the ball correctly, and twenty-three, the cube. The children had less accurate knowledge of the parallelepiped (beam). Only 16 out of 24 selected the parallelepiped. The others selected instead: five, a cone; two, a cylinder; and one, a pyramid. The pupils of the lower grades did not know cylinder, cone, or pyramid.

Let us examine the results of the investigation conducted in the upper grades of the auxiliary school. Asked to find a square, most pupils of the upper grades did it correctly; only one fifth grader and three seventh graders selected the rectangle. The lack of review of the geometric figures in the seventh grade and the sketchy study of them in the sixth grade raise the number of mistakes.

When we asked the children to select a rectangle, we observed an anomalous pattern. One fifth grader and three seventh graders selected the square, and the others solved the problem correctly. Selecting triangle and circle gave the pupils of the upper grades no difficulties. All pupils carried out this task correctly.

The selection of geometric bodies showed the following pattern. Not all pupils could manage to select the parallelepiped. Seventh grader R. M. selected a cone instead, although she had named the parallelepiped correctly in a previous series (though she did make a mistake in that). This indicates an uncritical attitude towards the operation. Some fifth and sixth graders chose the cylinder and pyramid. Selecting the cube caused fewer difficulties; only one fifth grader chose a cone instead. All selected the ball. The pupils of the upper grades encountered significant difficulties in selecting the cone, cylinder, and pyramid. More than half of the pupils chose the cone correctly; the others, for the most part, selected simpler figures instead, less familiar ones in particular. One fifth grader selected the parallelepiped, two pupils of the fifth and sixth grades selected the cylinder, and four pupils of the sixth and seventh grades selected the pyramid. In general, the pupils confused the cylinder with the cone and the pyramid. Almost half the pupils correctly selected the cylinder, the others selecting the cone or pyramid instead. The frequent selection of the pyramid was because the cone, in individual children's conceptions, is a geometric body that has edges. The pupils were accustomed to seeing a "cone" as a part of a hammer.

They also confused the pyramid with cone and cylinder. Only half the pupils selected the pyramid correctly. In selecting cone, cylinder, and pyramid, the children used the method of exclusion. Because of the lack of concise knowledge of the given geometric bodies, their choice was limited by unfamiliar geometric forms.

Answers to the questions, "Why does the pupil think that this is such-and-such a figure?" or "Tell all that you know about this geometric body" show the development of the child's logical thought. From the pupils' answers it is possible, to a certain extent, to judge the process of the formation of concepts of abstract geometric forms.

In the first grade the pupils gave simple answers for the definition of a circle; in some the distinctive features of the given geometric figure were not noted. Most answers were devoid of any thought, for example: "something drawn," "this figure," and the like.

In defining a circle the third and fourth graders remarked in most cases: "because it is round" or "it has no angles, but is all round." The third grader P. K. juxtaposed the circle to the square; he sharply distinguished these two figures. His answer was: "Because it has no ends, like a square, but is always round."

The triangle, too, has only one essential feature, and it was relatively easy for the pupils to give a verbal definition of the given geometric figure; that is, they expressed the elementary features of the given figure in verbal form. In the first grade the pupils did not answer the question posed them correctly. Only one pupil, B. Yu., answered, "Because it is three-angled, one above and two below." This pupil still cannot express his thought accurately in verbal form, but he already has, in elementary form, a concept of the triangle as a figure of three angles. Some third graders and most fourth graders answered that the given figure has three angles.

The pupils of the auxiliary school have greater difficulty in verbally defining the rectangle and square. These figures each have three essential features. The mentally deficient children of the lower grades cannot grasp all three features at once. Not considering the first and second grades, most third and fourth graders single out in their definitions only one essential feature: either "right angles" or "the sides are equal" (for a square) and "opposite sides are equal" (for a rectangle). In calling a rectangle or square a "quadrangle," the pupils emphasize that it has four angles. This shows that the mentally deficient child forms the elementary geometric concepts in verbal form only towards the fourth grade.

It is yet more complicated with the description of geometric bodies. Most children have difficulty telling about a given geometric body and more easily answer the question, "Why do you think that this is a ball?" Hence, during the investigation we had to resort to one of two forms of pupils' answers--either to a description of the geometric form or to an answer to the question of why he thinks that this is a ball or a cube.

In most cases the verbal description of geometric bodies with all the essential features is beyond the powers of mentally deficient schoolchildren. Most pupils described a ball as a figure having a round form. But some pupils defined it thus: "One can play with it," "one can roll it," "this is a playing ball"; that is, in their definitions they noted particular qualities characteristic of this geometric form and connected with the child's previous experience. In the lower grades the pupils of the auxiliary school attributed the qualities of similar objects to a geometric form. To a great extent this dealt with geometric bodies.

The children encountered significantly more difficulties in describing a cube. For many children a cube is called a cube because it is "four-angled." For some children "this is a little jar, in which one puts ointment." In describing a cube, the children ascribe to it the qualities of objects recalling a cube by their form. We observed the same phenomenon in describing a parallelepiped. Some children remarked that one could sharpen knives with a beam. In describing cone, cylinder, and pyramid the children noticed qualities of objects which they called by that name.

For pupils of the auxiliary school, describing geometric bodies presents more difficulties than does definition of geometric figures.

The difference between the pupils of the auxiliary school and of the public school is especially obvious in their descriptions of geometric forms, that is, when interpretation of the forms is required, especially an abstract idea of some particular geometric form. As examples, let us present the answers of pupils in different grades.

To the question "Why do you think that this is a square?" the pupils of the auxiliary school answered:

K. S., first grade girl: "It is drawn."

V. L., fourth grade girl: "Because it has right angles everywhere equal."

R. M., seventh grade girl: "Because it has four sides all equal."

These same pupils described a cube thus:

K. S.: "It is made to pack things in."

V. L.: "Because it has four sides."

R. M.: "A cube has four sides, all four sides are equal; these are angles, these edges, these sides."

Unlike the pupils of the auxiliary school the pupils of the public school, as early as the first grade, noticed the distinguishing feature of those figures which they selected. In the fourth grade of the public school, the pupils already gave accurate definitions to geometric figures. For example: "Because in a square all angles are right and the sides are equal."

In describing a cube and a parallelepiped, the pupils gave the following answers: "A cube has six faces, twelve edges, and eight vertices. All faces are equal and have a square form." "A parallelepiped has six faces, twelve edges, and eight vertices. In it, the opposite faces are equal." And one pupil even added to this: "... There are parallelepipeds which have four faces equal, but this one has only two." In the fourth graders' answers there is a clear understanding of the basic distinctive features of the given geometric figures. Thus, in the description of geometric figures a sharp difference is observed in the construction of logical connections by pupils of the auxiliary and the public schools.

The formation of elementary geometric concepts is connected with a series of factors, one of which is the structure of the geometric form, and another of which is the child's previous experience. The ball and circle are less complicated geometric forms than the triangle. The children master these geometric forms more easily in the process of perceiving them and encounter no difficulties in the process of forming elementary geometric concepts of these figures and bodies. The next geometric form with respect to difficulty is the cube.

In the formation of elementary geometric concepts the child's previous experience also plays a great role. The children encounter little circles, balls and cubes not only in lessons, but, also in everyday life (playing ball and balloons have the form of a ball). Since early childhood the pupil has observed the given geometric form, and the impression is made more deeply. The same holds in relation to cubes, which children often use in games.

Mentally retarded children made more mistakes than pupils of the public school. The answers to questions also indicated a qualitative difference. Whereas the pupils of the public school defined geometric forms accurately, the pupils of the auxiliary school gave vague, indefinite answers, most of which were devoid of any thought. The difference between the pupils of the public school and those of the auxiliary school is observed mainly in the interpretation of geometric forms. The answers given by pupils of the upper grades of the auxiliary school show that they also are capable of mastering geometric material; but successful mastery requires specific methods and planned systematic work beginning from the first grade.

Ascertaining the Ability to Find the Corresponding Form from Given Geometric Form.

In investigating geometric knowledge of figures we proceeded in the following way: figures of various dimensions were arranged on a table: a circle, radius 5 cm ; a triangle, base and height 6 cm; a square, base 7 cm; and a rectangle, height 8 cm.

When a circle and triangle were presented, the children selected the appropriate figures. When a rectangle was demonstrated, the following mistaken answers were given: a third grader selected a circle, and one first grader and one fourth grader of the auxiliary school selected a square. When a square was presented, a first grader and a fourth grader selected a rectangle. In perceiving geometric figures some pupils of the auxiliary school confused the rectangle with the square and circle, and the square with the rectangle.

In the next assignment we complicated the material, asking the children to select a figure analogous in form to the one demonstrated from among collections of colors, red and blue, which had been set out. The dimensions of both collections were equal to the object introduced.

When a circle and triangle were introduced, all the pupils selected the appropriate figures, mostly from the blue ones although to a lesser extent from the red. Two pupils selected two figures of different colors. When a rectangle was introduced, first grader K. S. selected a square. The pupils of the auxiliary school sometimes select figures by their accidental features (color, dimension, but not form). Most often the pupils confused the square and the rectangle; this happens because there are no sharp distinctions between these figures as, for example, between a circle and a triangle.

In one of the assignments the conditions for solving the problem were significantly more complicated than in the previous one. On the table figures of three colors were arranged: red, green, and yellow. The figure being demonstrated by the experimenter was brown. For the most part the selection of figures depended on their placement. Color played a secondary role.

The figures were set out in the following order (Figure 9).

In most cases the pupils selected green squares, yellow triangles, and green rectangles. One first grade girl answered, "It is not here," and a fourth grader selected three figures. The pupil selected the figures of a certain color because it was more convenient to take them.

In the next assignment the number of figures set out was increased. Before the pupils were figures of eight colors (blue, red, brown, yellow, white, black, pink, and green). Moreover, the collection set

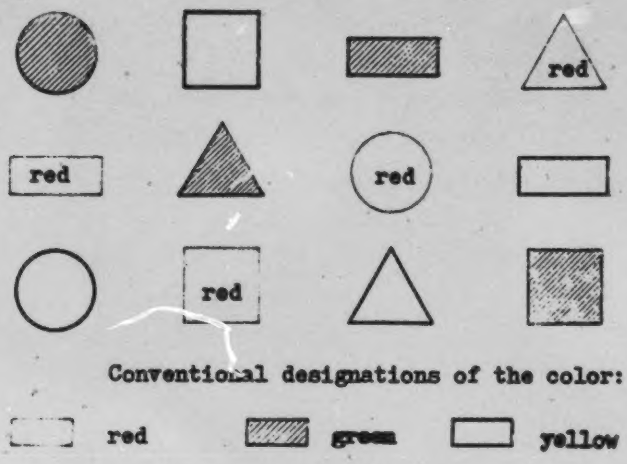

Figure 9

out included one complete set identical to the one demonstrated. More than one-half the pupils selected the triangle, rectangle, circle, and square equal in color and dimension to the model, then the figures having dimensions equal to those of the model but of different colors. The auxiliary school pupils made color and dimension the basis of their selection. When a square was presented in this assignment, only one pupil, first grader K. S., selected a rectangle; all the other pupils solved the problem correctly.

In a new assignment the pupils selected figures from a collection in which there were no figures identical to the one demonstrated. Most pupils chose yellow figures; that is, they maintained a constancy in the perception of color (the figures demonstrated had been yellow). Some first, second, and third graders confused rectangle and square; and one second grader answered, when a triangle was introduced: "There are none of these here." The pupils made more mistakes in this assignment than in the previous one.

In the next assignment the pupils were required to find, from a given geometric figure, all the corresponding ones. One can divide the pupils' responses into two fundamental groups. In the first group go the selection of blue figures, and in the second group the selection of all the given figures. In the first group, the dominant aspect in the perception of geometric figures is color. Form is the dominant aspect for the second group.

The children gave the best results when the circle was introduced; then the triangle; then the rectangle; and finally the square. Some pupils did not distinguish rectangle and square, just as in the previous tasks. The greatest number of figures selected were identical to the ones demonstrated. Then came figures near in dimension to the one demonstrated. Figures of smaller dimensions were selected by the children in a insignificant number.

Mentally deficient children spend much time solving this problem. But when they were asked "But did you choose all the figures?" most pupils answered "All," although some of the same figures remained in the collection. The children considered that they had solved the given problem correctly, and many did not look for the remaining figures.

In the series of assignments the pupils of the upper grades selected the appropriate geometric forms; but in one of the assignments two seventh graders, M. R. and I. Z., made mistakes: instead of a rectangle, one selected a square, and the second a triangle. In all the assignments color dominated in most cases. Sometimes however, the pupils selected figures of arbitrary color but of similar form.

When the pupils were asked to select all the geometric figures similar in form to the one introduced, the selection of circle and triangle caused the fewest difficulties. All pupils selected the circle with the exception of a sixth grader who chose a yellow square of much smaller dimensions from the collection. To the question "Why do you think that these are the same figures?" the pupils answered that "they are all equal" or "because they are all round" or "because they are all round and of the same color." The basis of the pupils' determinations was the similarity in form of the given geometric figures.

In the selection of the triangles, there were some cases in which not all the figures were selected. The pupils had the greatest difficulties in selecting the square and the rectangle. In selecting a square, a seventh grader chose two rectangles, instead, and a fifth grader, V. L., selected only two blue rectangles at first; but when he was asked "Did you chose all the figures?" he began to select squares and rectangles from the collection. After the above question the other pupils looked more attentively than the pupils of the lower grades, and chose the remaining similar figures.

In selecting the square the pupils did not equally select figures of different dimensions; for example, only half the pupils selected a green square (the smallest). In selecting the rectangle, fifth grader V. L. selected all the squares in addition to the rectangles. In the upper grades the dimensions of the figures, to some extent, also played a definite role in the selection of geometric figures.

Upon complicating the assignment, when the pupils were asked to choose figures without regard to color and dimension, the pupils made a series of mistakes; if the figures were sharply distinguished from each other as, for example, circle and triangle, then all the pupils selected them from the model without mistake. The auxiliary school pupils made mistakes when among the geometric forms similar ones were found, that is, a rectangle and a square.

The selection of geometric bodies proceeded somewhat differently. In one of the assignments all the pupils of the upper and lower grades of the auxiliary school correctly selected geometric bodies identical to the ones introduced. In this task a cube, a sphere, a parallelepiped, a cone, a cylinder, and a pyramid were set out on a table.

In the selection of geometric bodies we observed a certain difference from the analogous tasks with geometric figures. Whereas, in the investigation on ascertaining the knowledge of geometric figures, the answers were grouped according to color and form, in the investigation on ascertaining the knowledge of geometric bodies, all the pupils selected the geometric bodies similar only in form. The difficulty in the selection of geometric bodies depending only on dimension was smoothed out.

Whereas in the assignment with geometric figures only, two pupils selected the squares of the smallest dimensions, in the assignment on geometric bodies more than half the pupils selected the bodies of smallest dimensions. A geometric body is objective, a form in relief, and of three dimensions as distinct from figures which are plane forms.

The children more easily perceive and select bodies if they are similar to those demonstrated. In addition many pupils selected the geometric bodies remaining in the collection, while in the selection of figures this was not observed.

In one of the tasks in selecting geometric bodies the results obtained were significantly better than in selecting geometric figures in the analogous task. Almost all pupils selected all the appropriate geometric bodies: the most complete selection occurred when the ball was demonstrated, then the cube and parallelepiped. The pupils made more omissions in selecting cone, cylinder, and pyramid. The auxiliary school pupils still do not distinguish these geometric bodies sufficiently. Of the remaining geometric bodies more were selected when the pupils were asked, "Did you select all the figures?" Thus geometric bodies, since they were more in relief and had three dimensions, were more clearly perceived by the pupils than geometric figures.

In the process of selecting geometric forms according to a given model, we asked the pupils to prove that they had selected the required geometric figure or body. The questions "But by what are they similar?" or "Why do you think that this is the same figure?" evoked some difficulties, since the child had to build up definite logical connections in the process of answering.

In the first grade not all children answered this question correctly. Some answers were devoid of thought: "It is drawn." "Because they are little cubes." And when geometric bodies were presented, they answered: "These are little umbrellas for the rain." "This wears caps." With their answers the children emphasized the individual distinctive features and did not connect them with the question posed. And only a few pupils said, "They are equal." The pupils of the upper grades would say, "They are equal," or "This is a triangle and that is a triangle," "They are round," "They are round, they are equally made, these are little balls," and the like.

The pupils' answers to the question "Why do you think this is the same figure?" for the most part reduce to the following two groups: For geometric figures the answers were of this kind: "They are equal," "This is a triangle and that is a triangle," or "This is round and that is round." For geometric bodies the answers were of this kind: "They are equal."

For comparison let us examine the results of the investigation conducted in the public elementary school. In all assignments the pupils of the public school selected the geometric figures and bodies correctly, except for one assignment on geometric figures. A second grader responded "There are no such here"; and, when a square was introduced in another task, a weak first grader selected, along with squares, several rectangles, and when a rectangle was introduced, selected several squares.

The number of figures and bodies selected in the public school exceeds the number of figures and bodies selected in analogous assignments in the auxiliary school. This indicates a more critical approach by the public school pupils to fulfilling the assignment. When asked "Why do you think these are the same figures?" most pupils replied that "They are equal" or "This is a little circle and these are little circles," "They are also of the same form," "They are equal little balls, neither greater nor less," "This is a triangle and that is a triangle," and the like.

For the question "But have you selected all the figures?" the pupils looked among the remaining figures and bodies long and attentively and selected the additional ones. For the pupils of the public school color played a less essential role than for the auxiliary school pupils. This is because the pupils of the public school form more accurate concepts of geometric figures and bodies, whereas the auxiliary school pupils' concepts are more elementary and border on simple representations. For example, when two complete sets, equal in dimensions, were set out, one blue and the other red, the number of figures selected by the pupils of the public school were nicely divided into equal parts, but in most cases the pupils of the auxiliary school selected the blue figures. In another case only first and second graders selected figures equal in color and form; the third and fourth graders selected the figures similar in form which it was most convenient for them to take.

In selecting geometric bodies color plays an even less significant role than in selecting geometric figures. Unlike the pupils of the public school, the mentally deficient children have mistaken and imprecise perceptions of geometric forms. Most of the mentally deficient children begin the selection of figures by choosing those similar to the one introduced, according to their color and dimensions (of equal form). The pupils of the public school begin to select from the first figure they come across and choose ones similar in form.

In selecting geometric bodies the auxiliary school pupils did not follow the same pattern as in selecting geometric figures, but selected the first body they might find similar in form.

On the basis of the series of assignments we can conclude that the pupils of the lower grades of the auxiliary school have elementary geometric concepts. However, the children cannot yet give verbal expression to the elementary geometric concepts which they have formed. The pupils have elementary concepts of some geometric figures even in the first grade, but in most cases they are expressed in concrete form. When different figures or bodies of different sizes and colors were set out before a child and he was asked to select one or all the figures according to a model, even though he had no elementary geometric concepts of the given forms, he would nevertheless select figures identical to the model. In reality only in some cases did the pupils select figures of identical color. Even then when the child selected figures similar in color, the dimensions were not always equal. In all other cases the pupils selected figures similar in form but different in color and size. This fact indicates that even first graders can form elementary concepts about a given geometric form.

The most complicated process, that of expressing elementary geometric concepts verbally at a given stage of instruction, causes many difficulties and takes place in the auxiliary school only towards the fourth grade. Besides, defining a geometric form also depends on the number of essential attributes. The fewer the essential attributes which some particular geometric form possesses, the easier it is for the mentally deficient child to define it.

Comparing the results of the investigation of the individual assignments, we can remark that the process of fulfilling assignments does not proceed equally.

In analyzing the average results of the correct solutions in the lower grades of the auxiliary school, we note that the pupils of the auxiliary school easily select geometric figures and bodies according to a model. The mistakes that occurred when the pupils were given the name and were asked to select the geometric forms. And the pupils experienced significant difficulties in naming geometric forms independently. In all these tasks better results were obtained for geometric bodies than for figures.

There is not enough instruction in naming geometric forms. The pupils themselves have difficulty naming geometric forms. When the teacher names them, the pupil more easily connects the name with the form. A difference is also observed in the process of drawing figures and models of geometric bodies. The children represent geometric figures more easily and have difficulty in modelling a geometric body. The auxiliary school pupils more easily reproduce geometric forms when given a model than when given a name.

Although the program of the auxiliary school provides for classes in modelmaking, not enough of this type of work is provided. In one of the assignments the pupils of the auxiliary and of the public school made representations of a cube, parallelepiped, and cylinder. The task consisted of two parts: first the representation was constructed before the model was demonstrated, and second the pupils made the representation after the model was demonstrated.

None of the pupils of the lower grades of the auxiliary school could make a representation of a cube and parallelepiped correctly before this was demonstrated. In the upper grades only two fifth graders made a development of the cube and parallelepiped correctly, and almost all seventh graders did it correctly.

Significantly more pupils made a representation of the cylinder correctly. (Even some pupils of the lower grades made a representation of a cylinder; for example a third grader and a fourth grader did it correctly.) In the upper grades three fifth graders, three sixth graders, and four seventh graders made a correct representation of a cylinder.

For the most part the pupils of the lower grades represented a cube and a parallelepiped by means of tracing and cutting out only one face, and in making a representation of a cylinder they cut out only one base. Some pupils represented a cube and a parallelepiped by cutting out the six faces separately, and a cylinder by cutting out its two bases. Some pupils represented cube and parallelepiped in the form:

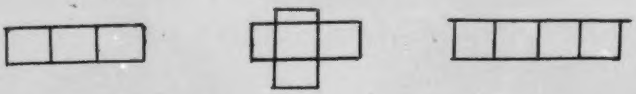

Figure 10

and the like; that is, they cut out three, four, and five faces. Many pupils could not carry out this task.

The pupils of the upper grades, especially of the seventh, in most cases made the representation correctly. The mistakes made by pupils of the upper grades in making the representation were the same as those of the lower grade pupils; that is, they cut the representation out according to one face, or according to six faces separately, or three, four, and five faces. Half the sixth graders did not know how to carry out this assignment.

After the demonstration the pupils of the lower grades made all three representations correctly. The greatest number of correct constructions were in representations of the cube, cylinder, and parallelepiped.

One first grader, K. S., even after the demonstration cut out all three representations in the form of bands, and second grader G. V. traced eight squares and cut them out as a whole. After the demonstration, the pupils of the upper grades made all three representations correctly.

Two weak seventh graders, R. M. and Yu. N., made representations correctly during the lesson, but in the individual investigation before the demonstration of the model they made mistakes, and only after the demonstration did they do it correctly. How to represent a cube, a cylinder, and a parallelepiped was taught only in the seventh grade.

As we can see from the investigation, after the demonstration even the first graders carried out this task correctly. For comparison let us examine the results of the investigation conducted in the public school. Before the demonstration only first graders could not carry out this task correctly. For the most part they cut out one face for representations of the cube and parallelepiped, and one base for the representation of the cylinder. One pupil cut out six faces separately for the cube and parallelepiped and two bases for the cylinder, and one pupil did not fulfill this task at all. Before the demonstration half the second graders made these representations correctly. The rest either cut out one face (a square for the cube, a rectangle for the parallelepiped, and a circle for the cylinder) or did not know how to complete this assignment.

In the third grade the results were more favorable. One pupil did not make the representations of the parallelepiped and cylinder, and two pupils did not complete the representation of the cube. The other pupils managed this task well. All the fourth graders carried out the task accurately.

After the demonstration, without exception, all the pupils of the lower grades of the public school carried out this task correctly. We see accuracy and confidence in the work of the pupils of the lower grades of the public school and the lack of accuracy and precision in the pupils of the lower grades and even in some pupils of the upper grades of the auxiliary school.

Summing up what has been stated above, we conclude that:

a) The pupils of the auxiliary school easily select a form accordingly to a model. The increase in difficulty depended on the supplementary features. Upon complicating the assignment by asking the pupils to select figures without regard to color and dimensions, the pupils of the auxiliary school made mistakes. If the figures were sharply distinguished from each other as, for example, circle and triangle, then all the pupils selected them according to a model without mistakes. The pupils of the auxiliary school made mistakes among the similar geometric forms, such as a rectangle and a square.

b) The next stage of difficulty in the study of geometric forms is their selection according to name, the representation of geometric figures in a drawing, and the modelling of geometric bodies.

c) The pupils of the auxiliary school encountered more difficulties in naming geometric forms than did the pupils of the public school. At first the child does not master geometric concepts through precise verbal formulation characterizing the geometric form, but by means of concrete comparison with some object familiar to him from his daily life. For example, the cylinder is a little jar, a pipe, a little barrel, and the like.

d) The pupils of the auxiliary school have diverse ideas about geometric forms. They have a better idea of the solid form than of the plane form. At the same time they do not have equal ideas of solid form; they have a better idea of ball and cube and a less accurate idea of others. Of the plane forms they have a better idea of circle and triangle and a poorer idea of square and rectangle.

e) More is required of the auxiliary school pupils than of the pupils of the public school, especially in organized studies, since in the process of practical activity these pupils master abstract geometric concepts with greater difficulties. Hence one must conclude that studying geometric material should occupy an especially prominent place in an arithmetic course.

f) The auxiliary school pupils are capable of mastering elementary geometric ideas and concepts.

An Investigation of Auxiliary School Pupils' Knowledge of Square and Cubic Measures

The study of square measure in the auxiliary school occurs in the sixth grade as stipulated by the program. In the public school this topic is studied in the fourth grade. For this investigation we selected six sixth graders and six seventh graders of the auxiliary school and six fourth graders from the public school. The investigation was conducted in the fourth quarter; by this time square measures had been studied.

The following different types of problems may be encountered in the process of studying square measure:

a) computing the area of a rectangle or square of a given geometric form;

b) computing the area of a rectangle or square from the dimensions of given figures;

c) computing the area of a rectangle and of a square in a concrete situation (this problem is a practical application of the first case);

d) computing the areas of rectangular or square sections or lots of given dimensions (this problem is a practical application of the second case).

Besides the cases indicated above, more complicated problems may be encountered: for example, computing the total area of a figure composed of several squares and rectangles. In the work process another problem may be encountered: the task of tracing a geometric figure of given dimensions and calculating its area. The pupils become acquainted with the drawing of geometric figures while studying geometric forms.

We made it our goal to ascertain the knowledge, abilities, and skills of the pupils of the auxiliary and public schools with respect to square measures and the solution of problems in calculating areas. As material for this investigation we took three rectangles and two squares of different dimensions which were traced out on a paper.

The dimensions of the rectangles were as follows:

a) length 12 cm , width 2 cm;
b) length 12.5 cm , width 7 cm;
c) length 9.5 cm , width 8.5 cm.

The dimensions of the squares were: 1) 6 cm and 2) 7.5 cm.

The assignments were given to the pupils in order of increasing difficulty. In determining the dimensions of the rectangle the pupils encountered three cases:

a) The sides of the rectangle were expressed by whole numbers of the basic units of measuring.[3]

b) One side of the rectangle is expressed by a whole number of the basic unit of measure and the second side is expressed by a fractional number of the basic unit of measure or by a composite concrete number.

An attempt to ascertain the pupils' abilities to determine the area of a rectangle and of a square and to solve concrete problems was conducted in a second series.

The following problems were proposed to the pupils:

a) "What is the area of a rectangle whose length is 7 cm and whose width is 3 cm?"

b) "What is the area of a square having a side of 8 cm?"

c) "What is the area of a Pioneers' room if it is 7 m long and 6 m wide?"

d) "What is the area of a three room apartment if the length of the largest room is 6 m and its width is 5 m, the length and width of the middle room are each 4m, and the smallest is 3 m long and 2 m wide?"

1. The Pupils' Knowledge of Square Measure

Let us examine the results of the investigation on the pupils' knowledge of square measure.

Before determining the area of a figure, a pupil independently measured its length and width. In solving this and the following problems the pupil who carried out the task independently was given complete freedom.

Determining the dimensions of a rectangle 12 cm long and 2 cm wide in the sixth grade of the auxiliary school caused no difficulties.

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[3] By basic units of measuring we mean those which for the given concrete case express a whole number of units which does not exceed the next larger unit. And in this case there may be some exceptions. For example, the length of a rectangle is 12 cm; this would be written as 1 dm 2 cm, but in practice we used centimeters more often than decimeters.

All the pupils determined the dimensions correctly and correctly computed the area. Yet one question arises, the most complicated one for the pupils of the auxiliary school--this is the form of writing down the units. Before the experiment had begun, all pupils were told to write down the solution correctly. Three ways of writing down the solution of the given problem are possible:

1) 12 sq. cm x 2 = 24 sq. cm
2) 12 x 2 = 24 (sq. cm),
3) 12 cm x 2 cm = 24 sq. cm.[4]

No sixth grader wrote down the solution of the given problem correctly.

Two pupils wrote it down in the form: 12 x 2 = 24 sq. cm. One pupil, who had measured the sides of the rectangle correctly, wrote down the solution in the form:

12 x 2 = 24 sq. cm.

There were other ways in which notations were made:

12 cm x 2 cm = 24 cm ,
12 cm x 2 = 24 cm ,
12 cm x 2 = 24 cm.

In the first three notations, mistakes are found only in the form of the notation. But the final result is written down correctly. In the notations immediately above, the solution is devoid of thought. These pupils still did not understand the essence of computing areas. The pupils mechanically recalled that for determining the area of a rectangle one must measure the length and width and multiply the numbers obtained, but they still had not mastered the essence of measuring an area. The pupils still did not differentiate between linear and square measures. It was all the same to them in whatever measures one expressed the area of a rectangle, as long as the calculations were correct.

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[4] The contemporary methodologists N. N. Nikitin, A. S. Pchelko, and others use the following form of notation in the elementary school: 12 sq. cm x 2 = 24 sq. cm. In the auxiliary school it is also advisable to have only one mode of notation.

The next figure whose area was to be found was a rectangle 12.5 cm long and 7 cm wide. In this problem the pupils encountered new difficulties—a precise determination of the dimensions. Only three of the six pupils determined the dimensions correctly, and three pupils rounded them off to whole numbers--instead of 12.5 cm they wrote down only 12 cm. The pupils who had determined the dimensions correctly made mistakes, in multiplying the two numbers: from 12 1/2 x 7 they got 84 1/2 sq. cm or 84 1/4 sq. cm. The erroneous results of the calculations are due to the lack of a definite algorithm for multiplying a common fraction by a whole number, although the sixth graders had already studied a section of this topic. Thus no sixth grader determined the area of the rectangle correctly.

Determining the area of a rectangle whose dimensions were 9.5 cm and 8.5 cm called forth more difficulties. In determining the dimensions there were almost the same mistakes as in the previous case: three pupils determined the dimensions correctly, two pupils rounded them off to whole numbers, and one pupil measured only one side, having taken the given rectangle to be a square. The pupils who determined the length and width correctly made mistakes in multiplying.

The following results were obtained from three pupils:

1) 9 1/2 x 8 1/2 = 72 1/2 sq. cm;
2) 9 1/2 x 8 1/2 = 72 1/2 cm;
3) 9 1/2 x 8 1/2 = $\frac{17}{2}$ = 8 1/2.

The curriculum for the sixth grade of the auxiliary school does not stipulate multiplication of a common fraction by a fraction. To get the correct solution in the given case, it was necessary to change centimeters to millimeters. No pupil did this. The pupils did not even begin to think about the question of whether they could multiply a fraction by a fraction or whether they had studied this in class. Such an approach to solving the problem characterizes the peculiarities of the mentally retarded child's psyche as distinct from that of the public school pupil. This peculiarity consists in the absence of a critical attitude toward one's actions.

Instead of the area one pupil determined the perimeter, and wrote his final result in square meters; this was also his form of solution for subsequent problems on determining the area of a square. The pupils' form of notation remained constant for the most part, i.e., the same as in the previous problems. Only one pupil in the first case wrote down her answer thus:

12 cm x 7 = 84 cm ,

and in the second case:

8 1/2 x 9 1/2 = 72 1/2 cm.

The auxiliary school sixth graders could solve such a problem completely if the centimeters were changed into millimeters, or if the dimensions were all measured in millimeters, but no pupil did this.

In determining the area of a square we took two possible cases: 1) the length of a side of the square is 6 cm, and 2) the length of a side of the square of 7.5 cm ; that is, the length of the side is expressed in whole numbers and in fractions. Computing the area of a square with a side of 6 cm produced almost no difficulties; 5 pupils out of 6 calculated the area correctly, but one pupil calculated the perimeter of the square. The form of the pupils' notation was just the same as in calculating the area of a rectangle. Three pupils wrote down 6 x 6 = 36 sq. cm , and the others wrote 6 cm x 6 cm = 36 cm ; 6 x 6 = 36; 6 cm x 6 = 36 cm.

In determining the area of the square with side 7.5 cm long, the pupils encountered difficulties, not so much in determining the dimensions, as in calculating. The pupils who determined the length of a side as 7 1/2 cm, upon multiplying 7 1/2 x 7 1/2 made various mistakes: 7 1/2 x 7 1/2 = 49 1/2; 7 1/2 x 7 1/2 = 14 1/2; and 7 m 5 mm x 7 m 5 mm =50 m.

Although pupil K. T. did write down the dimensions in the form of a composite concrete number, he did not put down the units correctly and he did not break down the measurements into smaller units.

For the most part the form of notation remained the same as in the previous problems. Two pupils wrote down the solution as 7 1/2 x 7 1/2 = 49 1/2 sq. cm ; and three pupils as 7 cm x 7 cm = 49 cm ; and one pupil wrote 7 1/2 x 7 1/2 = 14 1/2 = 7.

In ascertaining the sixth graders' abilities to calculate the area of a geometric figure from drawings, the following insufficiencies in the pupils' knowledge were uncovered:

1) When the children are given the task of calculating the area of a figure whose dimensions are expressed by composite concrete numbers or by fractional numbers, the pupils cannot make the calculation correctly, not to mention writing down the solution in precise form. But in practice such cases are exactly those most often encountered.

2) Sixth graders do not know how to write down the solution of a problem correctly. Among the mentally deficient children a state is often encountered in which the pupils do not distinguish linear and square measures in solving problems. The most diverse forms of notation are encountered. In the sixth grade there was not one correct notation. For example: 8 1/2 x 9 1/2 $=\frac{17}{2}$ = 8 1/2; 12 cm x 7 cm = 84 cm ; 8 1/2 cm x 9 1/2 cm = 72 1/2; and the like.

Let us examine the results of the seventh graders of the auxiliary school in calculating the areas of a rectangle and of a square from drawings. A rectangle with sides of 12 cm and 2 cm was determined correctly by four out of six pupils. More difficulties were encountered in determining the area of a rectangle of dimensions 12.5 cm and 7 cm. Only three out of six pupils determined the dimensions correctly. In this case the notation was also marked by the absence of uniformity; for example, 12.5 x 7 = 87.5 sq. cm ; 12.5 x 7 = 87.5 cm ; 12 1/2 x 7 = $\frac{25}{14} = 1\,\frac{11}{14}$.

The pupils did somewhat better in determining the area of a rectangle of dimensions 9.5 cm and 8.5 cm. Four pupils determined the dimensions incorrectly, and M. R. determined the perimeter in addition. One pupil determined the dimensions correctly, but when she began to write down the result in decimal fractions, she made an error. Instead of writing down 9 1/2 sq. cm x 8 1/2 = 9.5 sq. cm x 8.5, she wrote down 9.2 sq. cm x 8.2. Changing common fractions into decimals caused an erroneous answer.

Five out of six pupils determined the area of a square with a side of 6 cm correctly. The notations were diverse: 6 cm x 6 cm = 36 sq. cm ; 6 cm x 6 cm = 36 cm ; 6 x 6 = 36 cm ; 6 sq. cm x 6 sq. cm = 36 sq. cm.

It was a somewhat more complicated matter for them to compute the area of a square with a side of 7.5 cm. Only four out of six pupils determined the dimensions correctly. For the most part the notation remained the same as in the previous case, with the following changes. Instead of the cases 6 sq. cm x 6 sq. cm = 36 sq. cm ; and 6 cm x 6 cm = 36 cm ; there appeared notations of the form: 7.2 x 7.2 = 51.84 and 7.5 cm x 7.5 = 56.25 cm.

Ascertaining the Pupils' Ability to Determine the Area of a Rectangle and a Square When the Dimensions are Given

Four problems were given the pupils. The basic aim of these problems was to ascertain their ability to determine the areas of figures from given dimensions. In the first problem, "What is the area of a rectangle 7 cm long and 3 cm wide? " all the sixth graders of the auxiliary school gave a correct answer — 21; but in every case there was a mistake in notation; the answer was 21 cm. In their notations the sixth graders confined themselves to the forms: 7 cm x 3 cm = 21 cm and 7 cm x 3 = 21 cm.

In the second problem, "What is the area of a square whose side is 8 cm long?" only four pupils made the correct calculation, 8 x 8 = 64; and, for the most part, the form of the notation remained the same as in the first problem; that is 8 cm x 8 cm = 64 cm or 8 cm x 8 = 64 cm. Instead of computing the area, two pupils computed the perimeter of the square, one of them computing the perimeter in square units: 8 x 4 = 32 sq. cm.

The third problem, "What is the area of a Young Pioneer's room if it is 7 m long and 6 m wide [7:77]?" did not produce special difficulties either. All the sixth graders made the computation correctly, but the form of the notation remained as before--42 m was obtained as an answer. Only one pupil wrote down 7 x 6 = 42 sq. m. Here, too, there was imprecision in the units, but the final result had a definite geometric sense.

In the fourth problem, "What is the area of a three-room apartment if the length of the largest room is 6 m and its width is 5 m , the length and width of the middle room are each 4 m , and the length of the smallest is 3 m , and its width is 2 m [7:78]?" Four sixth graders determined the area of the apartment in linear measurements. The notations were: 1) 6 m x 5 m = 30 m ; 2) 4 m x 4 m = 16 m ; 3) 3 m x 2 m = 6 m ; 4) 30 m + 16 m + 6 m = 52 m.

One pupil solved it thus: 1) 5 x 6 = 30 sq. m ; 2) 4 x 4 = 16 sq. m ; 3) 3 x 2 = 6 sq. m ; 4) 30 + 16 + 6 = 52 sq. m. Although in this solution the partial and final results do have definite sense, the units were handled imprecisely. And one pupil confused all the data, writing down the solution thus: 1) 6 m x 3 = 18 m ; 2) 5 m x 3 = 15 m ; 3) 18 x 4 = 72 m ; 4) 15 x 3 = 45 m ; and 5) 6 x 2 = 12 m. She multiplied the dimensions of one room by the dimensions of another.

Although the pupils encountered more difficulties in solving the practical problem, they solved the problems more correctly and quickly when computing the area from definite data. This is because the pupils, in order to determine the area of a figure from a drawing, first had to determine the dimensions, and then compute the area. The approach to solving problems of this category is more creative than the approach of the second category, in which the pupils determined the area of a figure from its dimensions. In this case the pupils often multiplied the length by the width mechanically, without considering the sense of the solution. In witness to this are the answers given by the pupils who determine the area in linear units.

The seventh graders' solutions of problems could be distinguished from the solutions of analogous problems by sixth graders. The seventh graders gave more accurate notation. Half of the pupils wrote the solution down in the form 7 cm x 3 cm = 21 sq. cm, and the rest wrote the solution in the form 7 cm x 3 cm = 21 cm. These two notations alone were characteristic of the seventh graders. These pupils encountered some difficulties in computing the area of a square; three pupils computed the perimeter, one of them in sq. cm ; and, in computing the area of a rectangle, only one pupil calculated the perimeter instead of the area. All six pupils solved the third problem correctly, but only two had the correct notation; three wrote it in the form 7 m x 6 m = 42 m , and one in the form 7 x 6 = 42 m. The solution of the fourth problem caused more difficulties. The first three questions (calculating the areas of separate rooms) were determined by five pupils correctly; of these, only two wrote down the solution correctly. The fourth question, determining the area of the whole apartment, was correctly computed by four pupils; of these two wrote it in the correct form, that is, 30 sq. m + 16 sq. m + 6 sq. m = 52 sq. m. One pupil, instead of adding together the results obtained from the first three questions, multiplied these numbers and wrote the final result in square meters.

In this series we observed in the pupils a more accurate form of notation than in the first series. Greater uniformity was observed in individual pupils' notations. After the study of square measures we conducted a control work-experiment in the Auxiliary School No. 77 of Moscow and the Ul'yanovsk Auxiliary School No. 39. The pupils were asked to solve two problems, which we also proposed in individual studies. A characteristic aspect of their solution was more accurate notation, and more precise computation. All thirteen pupils of both Auxiliary School No. 77 and the Ul'yanovsk School No. 39 solved the following problem correctly: "What is the area of a Pioneers' room if it is 7 m long and 6 m wide?" Twelve pupils of the 77th School wrote the solution in the form 7 m x 6 m = 42 sq. m , and one pupil write it as 7 m x 6 = 42 sq. m. In Ul'yanovsk Auxiliary School No. 39 there were three forms of notation. Five pupils wrote down the solution as 7 x 6 = 42 (sq. m), and two pupils as 7 m x 6 m = 42 m.

In solving the more complicated problem "What is the area of a three room apartment if the largest room is 6 m long and 5 m wide, the middle room is 4 m long and 4 m wide, and the smallest is 3 m long and 2 m wide?" some diversity in the notation of the units was observed. The computations in both the Ul'yanovsk and the Moscow School were made precisely, but in the Ul'yanovsk School diverse notations were observed in determining the area of all three rooms. Along with the correct form of notation, 30 sq. m + 16 sq. m + 6 sq. m = 52 sq. m and 30 + 16 + 6 = 52 (sq. m), some pupils wrote: 30 sq. m + 16 sq. m + 6 sq. m = 52 (sq. m); 30 m + 16 m + 6 m = 52 m ; 30 m + 16 m + 6 m = 52 sq. m ; and 30 sq. + 16 sq. + 6 sq. = 52 sq. m.

In solving the problems some pupils formulated the questions imprecisely. R. M. made an essential mistake in formulating the question; instead of writing down "What is the area of the room?" he wrote "What is the width of the room?" R. M. did not distinguish such geometric concepts as area, width, and perimeter accurately enough. R. Kh. made a mistake in copying the conditions of the problem; instead of 6 meters she wrote down 6 centimeters. As a result of this mistake the problem made no sense, for there are no rooms of such dimensions. But R. Kh. did not consider the conditions of the problem and did not correct her error. In formulating the question N. K. also made a mistake; instead of writing down "What is the area of a three-room apartment?" she wrote "What is the area of three apartments?" N. K. did not make an accurate analysis of the given problem. In determining the area of the Pioneers' room, L. M. wrote down: "What is the volume of the room?"

In determining the areas of rectangles and squares the pupils of the auxiliary school made mistakes, but the nature of these mistakes was different for individual pupils. Whereas some pupils made mistakes of a semantic character (for example, they formulated the questions imprecisely; instead of the area they determined the volume or perimeter, not grasping the sense and meaning of the problem), other children made mistakes which we can ascribe to insufficient concentration of attention (for example, instead of meters they wrote down centimeters).

One more fact should be noted. In some cases in one assignment a pupil made a correct computation and a correct form of notation, and in another, similar task, the form of notation was erroneous. This fact indicates an incomplete mastery of the problem being solved.

Unlike the pupils of the auxiliary school, the pupils of the public school determined the dimensions more precisely, and operated better with composite concrete numbers.

Let us draw some brief conclusions:

a) The auxiliary school pupils are weak in their solution of problems in computing the areas of geometric figures from a drawing. More difficulties occur in determining the dimensions of figures whose length is not expressed by a whole number of the basic units of measure. In these cases the pupils usually round off the dimensions to whole numbers.

b) The pupils encounter fewer difficulties in determining the areas of geometric figures when the dimensions are given.

c) In most cases the pupils of the public school give an accurate and precise notation and understand the solution of the problem well, although with weak pupils some mistakes are encountered.

2. Investigation of the Auxiliary-School Pupils' Knowledge of Cubic Measure

To ascertain the pupils' abilities to determine the volumes of geometric bodies, we conducted two series of investigations. An individual investigation was conducted in the seventh grade of the auxiliary school and in the fourth grade of the public school.

To ascertain their ability to compute the volume of a parallelepiped from a given geometric model, we asked the children to compute the volumes of the following bodies:

a) a parallelepiped 6 cm long, 4 cm wide, 10 cm high;
b) a parallelepiped 4 cm long, 4 cm wide, 10 cm high;
c) a parallelepiped 2 cm long, 2 cm wide, 20 cm high;
d) a cube whose edge was 10 cm long;
e) a cube whose edge was 2 cm long.

The numbers representing the dimensions of these geometric bodies produced no difficulties. The parallelepipeds were of two forms. One had faces of rectangular form, and two had faces of square form; one of them had an elongated form (2 cm , 2 cm , and 20 cm). The pupils were required to be able to determine the dimensions of geometric bodies and to compute their volumes.

Some auxiliary school pupils had difficulty in determining the dimensions. Two pupils did not determine the dimensions completely. Of the three dimensions given, they determined only two. Moreover, they could not use these data correctly. One should also note that R. M. determined all the dimensions of the parallelepipeds and cubes correctly, but she could not use them in determining the volumes. In the first problem, instead of the volume, R. M. determined the surface area face by face, that is 6 x 6 = 36 sq. cm ; 10 x 4 = 40 sq. cm. Three pupils did the computation correctly.

For the most part the notations were of three forms: 10 cm x 4 cm x 4 cm = 160 cub. cm ; 10 cm x 4 cm =40 cm; 40 cm x 4 cm - 160 cub. cm. More often, however, one encountered notations like 10 cm x 4 cm x 4 cm = 160 cm. Only one pupil's notation was correct in all respects.

Determining the volume of a cube produced more difficulties. One pupil multiplied the length of an edge by the number of faces. In solving this problem there was no logic. In determining the volume of the cube one pupil even calculated its total surface area. After ascertaining the pupils' ability to determine the volume of parallelepiped and cube, given their dimensions, and to determine the volumes of objects having the form of a parallelepiped or cube, we asked the pupils to solve the following problems:

a) What is the volume of a rectangular parallelepiped 5 cm long, 3 cm wide and 8 cm high?

b) What is the volume of a cube whose edge is 7 cm long?

c) What is the volume of a classroom if it is 10 m long, 7 m wide and 5 m high?

d) There are eight patients in a hospital ward. How many cubic meters of air belong to each patient if the room is 8 m long, 6 m wide and 5 m high?[5]

Solving these problems caused fewer difficulties for seventh graders of the auxiliary school than solving the preceeding assignments. All the pupils computed the first problem correctly, but they did not use a correct form of notation. One pupil wrote it down correctly: 5 cm x 3 cm x 8 cm = 120 cub. cm. Three pupils wrote it as 5 cm x 3 cm x 8 cm = 120 cm ; one as 5 cm x 3 cm = 15 m , 15 x 8 = 120 cm; and one as 5 x 3 = 15 cub. cm, 15 x 8 = 120 cub. cm. The pupils gave analogous forms of notation and computation in determining the volume of the classroom. In this problem two pupils used a correct notation: 10 m x 7 m x 5 m = 350 cub. m.

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[5] We took the last two problems from [7:82], although we have changed the last problem somewhat--instead of sixteen people, we put eight.

The auxiliary school pupils sometimes make the computations mechanically and give little consideration to the meaning of the solution, a phenomenon analogous to what happened in the computation of areas. In determining the volume of a cube, just as in computing the area of a square, the pupils encountered more difficulties than in computing the volume of a parallelepiped. Only two pupils calculated correctly. One gave an incorrect answer; two pupils calculated the entire surface area expressed in cubic units; and one pupil determined the perimeter of a face in square centimeters. The auxiliary school pupils sometimes confuse certain concepts which they acquired in the process of studying geometric material, for instance volume and perimeter. As a result of this, the pupils determined the surface area or perimeter, instead of the volume. This is explained by certain difficulties experienced in determining the volume of a cube. The pupils do not have a good idea of the essence of determining volume, and rather mechanically use the methods of determination which they have studied. In determining the volume of a parallelepiped, the pupil more easily recalls that it is necessary to multiply length by width and by height than in determining the volume of a cube, in which one must repeat the length of the edge as a multiplier three times.

The children also encounter some difficulties in determining the volume of a parallelepiped in the more complicated problem in determining the cubic capacity of air in a hospital ward. For the first question all the pupils made a correct computation; but in determining the number of cubic meters of air per person two pupils made mistakes in the computation. For the most part the form of notation remained the same as in the previous cases. Three pupils wrote down the solution as 8 m x 6 m x 5 m = 240 m; three pupils expressed their answers in cubic meters.

One cannot ignore the fact that the very condition of this problem directs one to determine the volume in cubic meters, whereas in the previous problem the question was formulated as "What is the volume of a classroom." In this problem, in formulating the very question "How many cubic meters of air are there for each patient...?" it is indicated that the volume is to be determined in cubic meters and not in linear or square measure. Four pupils paid attention to this; but two pupils, even in this case, went on to determine the volume in square meters or without units at all.

To see how the subjects determine the volume of a room we conducted a control experiment in two classes, in the seventh grade of the Ul'yanovsk Auxiliary School and in the seventh grade of Auxiliary School No. 77 of Moscow. In Auxiliary School No. 77 the control work was conducted before the school material on calculating volumes was taught. In the work of these two grades a significant difference was observed in the notation of the units. The computation in the problem "What is the volume of a classroom if...?" were correct in both schools, but the notations differed. With the pupils of the Auxiliary School No. 77 one could find diverse notation; in the Ul'yanovsk school all the pupils computed the volume and wrote it down correctly: 10 x 7 x 5 = 350 (cub. m).

The second problem "In a hospital ward there are eight patients..." was harder for the auxiliary school pupils; they wrote down the most diverse notation for the units. Even among the correct answers there were several with erroneous notation; but in the Ul'yanovsk school all fifteen pupils wrote down and computed the first question. The notation of the second question was also more accurate in the 77th school: of thirteen pupils, ten wrote it correctly. In the Ul'yanovsk Auxiliary School the computation was correct, but the notation was of three forms: 1) nine pupils wrote it as: 240 cub. m ÷ 8 = 30 cub. m ; 2) four pupils as 240 ÷ 8 = 30 (cub. m); 3) two pupils as 240 cub. m ÷ 8 = 30 (cub. m).

Let us draw some brief conclusions:

a) Both in computing the area of a square and a rectangle and in computing the volume of a cube and a parallelepiped of given geometric forms, the pupils encounter significantly more difficulties than in determining the areas and volumes of these same forms when their dimensions are given. One can explain this by the fact that in the first case the pupils approach the solution more creatively, and in the second case more mechanically, lacking a sufficient number of exercises. Hence one must conclude that for better mastery of square and cubic measures, more lessons on determining areas and volumes of given geometric forms must be conducted. This work must be varied as much as possible, computing the areas of a table, the classroom floor, a room, a plot of land, and similar objects, or computing the cubic capacity of the classroom, of a room, and of various objects having the form of the given geometric bodies.

b) The second inadequacy in the pupils' studies is the absence of an accurate and correct form of notation in solving problems. This indicates that one must pay serious attention to correct notation in solving problems in computing areas and volumes.

c) Some pupils do not distinguish such geometric concepts as area, volume, and perimeter. These pupils often confuse one concept with another. One must given this problem serious attention as well.

d) Correct notations is perfectly accessible to the pupils of the auxiliary school, but one must give this problem the most serious attention.

Pedagogical Methods in Auxiliary School Pupils' Study of Geometric Forms

The study of geometric material is one of the integral factors in a person's cognition of the material world, and proceeds according to the general laws of the dialectic-materialistic theory of cognition.

The dialectic method of cognition of truth was accurately formulated by V. I. Lenin in his Philosophical Notebooks: "From live contemplation to abstract thought and from that to practice -- that is the dialectic way of cognition of truth, the cognition of objective reality [9:166]."

The child, beginning to be aware of the exterior world, turns to objects and phenomena through practical activity. Speaking of the child's thought, K. Marx indicated that

> ...his judgment as well as his practical thought have primarily a mechanical and sensory character. The sensory properties of objects and phenomena are the first bonds connecting him to the exterior world. Practical senses, primarily the senses of smell and taste, are the first senses with whose aid he appraises the world [9:166].

Recognition of the material world is possible only through sensations and perceptions. "Sensation is the image of moving material," wrote Lenin. "Except through sensation we can learn nothing about forms of matter or forms of motion; sensations are evoked by the operation of moving material upon our sense organs [10:288]." With the aid of the sense organs, supplemented by the activity of the memory and of thought, a person is capable of reflecting not only the separate properties and qualities of an object, but also the objects of the material world as a whole.

Psychological investigations have shown that perceptions, ideas, and concepts do not exist isolated from each other, but are a unified psychological process. Thus, for example, the immediate perception of objective reality with the aid of the sense organs is always supplemented to some degree by our conceptions, based on previous experience, concerning a given object. Simultaneously with this we acquire the capability of analyzing a given object and singling out its essential features. Geometric concepts are also acquired by contemplation (with the aid of perception and ideas) of objective reality. The process of abstracting is accompanied by the analysis of facts into individual features or signs (analysis) on the one hand, and by the combination of these elements into a unified whole (synthesis), on the other hand.

Perception of concrete form is accessible to children very early. The mastery of geometric form presents some difficulties at first. Children between three and seven years of age "...at first objectify" a geometric form, "That is, they give it a naive-objective interpretation: triangle--'little pocket,' circle -- 'little wheel,' quadrangle, cut through crosswise by intersecting lines perpendicular to each other--'little window,' a triangle placed atop a quadrilateral is 'a house,' and the like [11; see also 5:137-181]."

Then, in learning a geometric form, the child begins to perceive it as a similar form ("this is like a little window" and so forth). In the general course of the development of an objective and of a geometric form, a change is observed in the child: at first the child perceives the geometric form as proceeding from the objective; then, according to his degree of mastery of geometric representations and concepts, the concrete form of objects begins to be determined by means of abstract geometric forms. According to the degree of his familiarity with geometric material in the process of instruction, the child begins to distinguish definite geometric forms (ball, cube, circle, triangle, square, and so forth).

Let us now examine the ways of studying visual geometry in the auxiliary school, taking into account the peculiarities of the intellectual development of mentally deficient children.[6] The study of geometric material in the auxiliary school is stipulated by the educational curriculum, in which there are two basic stages. The first stage is the accumulation of geometric information and elementary ideas and concepts in the lower grades; the second stage is the systematic study of geometric material in the upper grades. Hence the ways of studying geometric material in the auxiliary school shall be examined according to these stages, that is:

1) The study of geometric material in the lower grades, and
2) The study of geometric material in the upper grades.

1. Methods of Studying Geometric Forms in the Lower Grades of the Auxiliary School

The scope of the geometric material is determined by the educational programs from the third grade upward. In the third grade the pupils familiarize themselves with geometric figures (circle, square, rectangle, and triangle) and with geometric bodies (ball, cube, and beam).[7] The pupils are able to distinguish these geometric forms from others, to find such forms in the environmental situation, to name them correctly, to outline the geometric figures by their contours, and to model geometric bodies from clay or plasticine. In the fourth grade the pupils are able to trace geometric figures--circle, square, rectangle, and triangle--by following their contours and also with the aid of a ruler, a compass, and set square. In addition, the program stipulates that, in the first and second grades, studies of geometric forms shall be conducted using various work methods: outlining, shading, and composing designs; cutting out and making models.

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[6] We will examine the study of geometric material at different stages of instruction in connection with our experimental data obtained as a result of our investigation of the pupils' knowledge of geometric forms.

[7] The auxiliary school curriculum provided only for the study of the right parallelepiped. In the future we shall refer to the right parallelepiped merely as a "parallelepiped."

Mentally deficient children come to the auxiliary school with an extremely poor store of geometric ideas. As our investigation showed, the spatial and geometric ideas of auxiliary school first graders are greatly limited and at times erroneous. The teacher's task is to make more accurate, and to put in good order, the pupil's store of geometric ideas, through systematic studies beginning in the first grade. In the first and second grades of the auxiliary school, studies of geometric material must also be conducted systematically.

On the basis of our investigation we propose the following order of studying geometric material.

The first step in the work is the study of geometric figures and bodies according to a model, and at this stage of work one must adhere to a definite sequence of the introduction and selection of geometric forms. This is especially important in the auxiliary school. On the one hand is the selection of geometric bodies and figures by form with dimensions and color identical to the one introduced. On the other hand is the selection of geometric bodies and figures by form with colors and dimensions different from those of the one demonstrated.

The second stage in the work is the classifying of geometric forms according to name.

The third and concluding stage is the pupil's naming of geometric forms.

The first stage in the work, as has been indicated above, is the selection of geometric figures and bodies according to a model. In our investigation all pupils correctly selected geometric figures and bodies similar in form and with dimensions and color identical to those of the model introduced. But when the pupils were asked to select from a collection in which there were figures of different dimensions and colors, they sometimes made mistakes.

The mentally deficient children have imprecise and diffuse perceptions. With such children it is especially necessary to conduct the work on selecting geometric figures and bodies from a collection with dimensions and colors identical to those of the model introduced. Then one must proceed to the selection of geometric figures and bodies similar only in form.

This stage of the work is initial in forming elementary geometric ideas and concepts. When a child selects a geometric form according to a model, he turns his attention to the essential features of that geometric form. At first the child cannot yet verbally analyze, according to these features, the specific properties of the geometric form; but during his practical activity, the child begins to establish, on the basis of previous experience, the similarity of two figures having the same form. When the child has concluded that objects having a solid round form are balls, he has already formed an elementary concept of the ball.

The selection of homogeneous forms according to a model should begin with the ball. The children have a better idea of this geometric form, since even before entering school in their games they have used objects or toys having the form of a ball (playing balls, little balls, apples, and the like). When guided by his previous experience, the child masters geometric forms more efficiently. The ball is the simplest geometric body and has the fewest essential features determining it. In our investigation we discovered that this geometric form was the most familiar to the pupils of the auxiliary school and of the public school. The form next most familiar to the pupils of the auxiliary school is the cube.

To familiarize pupils with geometric material, one should conduct with them a series of exercises in selecting geometric bodies and figures. These studies will help to distinguish geometric forms and to single out the elementary essential features.

In the process of every task the teacher must name the geometric bodies and figures correctly. The mastering of names will be gradual, but from the very beginning it is necessary to fix the pupils' attention on the correct name of the geometric form. The first impression which the child gets in the instruction process will be simpler and better consolidated in his mind. But if from the very beginning one gives the child imprecise names for the geometric forms, then later in the instruction process it will be significantly harder to teach him anew. At this stage of studying geometric forms one must conduct initial lessons which will promote a better assimilation of the material being studied. Then one should pass over to modelling geometric bodies from clay or plasticine.

The modelling of geometric forms should begin with the ball, by rolling the clay or plasticine between the palms of the hands.

It should be noted that modelling in the auxiliary school promotes a better mastering of geometric form. The cognition of geometric forms will become better and more meaningful only when we include more sense organs in the process of cognition. To the aid of vision and hearing come touch and the kinesthetic sphere. Correctly organized handicraft lessons are a powerful means promoting mastery of geometric forms. Without knowledge of geometric forms it is impossible to occupy oneself successfully with handicraft work; and without handicraft work lessons the study of geometric figures will not be as effective and will not promote correctional educational work sufficiently.

One should conduct work in the study of plane figures using a similar plan of operation. From a box of flat geometric mosaics, various geometric figures are chosen according to a given model. The selection should begin with the circle since it is the geometric figure most familiar to the children. At this stage a good educational tool is visual dictation, which develops the child's memory and thoughts.

Outlining geometric figures and also the bases of geometric bodies is a necessary stage in the process of instructing pupils of the auxiliary school. This form of work not only promotes a better mastery of geometric forms but is also a good preparatory measure for the study of writing. The auxiliary school pupils often have insufficient coordination of movements, especially of the hands. One should begin with outlining forms of larger dimensions in order to carry out the transfer to smaller ones afterwards, and finally to letters. So that the work might be more effective and interesting, after outlining one should conduct shading, coloring with crayons, and cutting out the figures drawn.

One should further conduct different assignments in cutting out and pasting together models. Without dwelling on this in great detail, we can only recommend to the teacher that he conduct various work with paper in handicraft classes [see 4].

The use of various table and movement games develops and widens the children's circle of elementary geometric ideas [see 8].

The next stage of work with the children is the selection of geometric forms according to name. Various projects in selecting geometric figures and bodies according to a model bring into a definite system the geometric ideas of the auxiliary school pupils and promote a correct mastery of the names of geometric forms. And it is possible to conduct a series of projects on this material in selecting geometric forms according to name. Work with a designer will supplement and widen the pupils' store of geometric ideas. All the previous work is basic to the third stage--the pupils themselves naming geometric forms. At this stage one should conduct projects on consolidating and widening ideas--modelling geometric bodies and drawing geometric figures according to name.

A comprehensive study of geometric forms in the first and second grades of the auxiliary school including modelling, outlining, shading, and making constructions promotes a better mastery of the geometric material.

We should always remember the words of the great Slavic pedagog John Amos Comenius:

> ...Let there be a golden rule for the pupils: allot everything, as much as possible, to the senses--the visible, for perception by sight; the audible, by hearing; odors, by smell; what can be tasted, by taste; and the tangible, by touch. If any objects are perceptible by several senses at once, let them be grasped at once by the several senses [1:207].

Each lesson should draw upon all of the senses to the maximum degree. Only then will the teacher reach his goal with the greatest success.

Studies of geometric material in the first and second grades must be conducted systematically. There may arise the practical question of how to study geometric material if there is no construction box in the school. In this case the pupils must make a series of geometric figures and bodies in handicraft classes. The geometric figures may be traced on cardboard, thick paper, on thin plywood and then cut out. The dimensions may vary from 1 to 6 cm with an interval of 1 cm. If figures of one centimeter will be too small and inconvenient, one may begin with dimensions of 3 cm. Then all the figures should be colored. The coloring should be as striking as possible, for in the lower grades the children show a greater interest in figures with striking colors.

The preparation of geometric bodies also presents no great difficulty. They may be turned out of wood if the school has a lathe, but if not they may be pasted together from thick paper by the children themselves. It is also advisable to color geometric bodies differently. The ball may also be prepared by the pupils. If it is impossible to turn balls out of wood, they may be modelled from clay or made from paper-maché.

If the pupils prepare geometric supplies under the teacher's direction, they will better consolidate and master the geometric material. We suggest that the pupils prepare the geometric supplies as much as possible. For this, the handicraft lessons may be used.

At the first stage of instruction we present the children with only one concept: the object.

Beginning in the third grade the child acquires some system of knowledge of geometric forms and some skills at reproducing them with the aid of modelling, tracing out a stencil, making cut-outs, and the like. This does not mean, however, that the auxiliary school pupils have already mastered the geometric forms. The children's mental functions, which are altered by their cerebral diseases, characteristically cause them to forget more quickly than the pupils of the public school; hence they need a greater number of repetitions in order to learn something.

It is useful at the beginning of the third grade to review what went on in the first and second grades, and then to start with a systematic study of geometric forms; it is not necessary that this section of the program be put off to the end of the year. Let us introduce an example of the study of geometric material in the 39th Ul'yanovsk School. The first and second graders had outlined and shaded geometric figures--circle, square, and rectangle. In the drawing lessons, geometric figures were proposed: these were the circle and square, and to a lesser degree the rectangle, and to an even lesser degree the triangle. In the third grade the study of geometric forms was assigned in the fourth quarter. Thus, after three quarters the third graders were not familiar with the geometric material (we were in the school in March, 1948).

In the process of individual studies with the pupils, we discovered that the first and second graders knew the geometric figures significantly better and drew them more precisely and more accurately than the third graders. Even a small complex of highly organized lessons on geometric forms conducted in the first and second grades showed significantly better results than lessons with the third graders. Assigning geometric material at the end of the school year is incorrect and uncalled for and should not be done in the practice of the auxiliary school. With these pupils it is necessary to systematically repeat the material being studied and to consolidate it more deeply, using various forms of work. Geometric material may be widely used within arithmetic lessons as didactic material in the study of computing. Even modest studies using geometric materials increase the pupils' knowledge significantly. But a system of specially selected exercises increases the pupils' knowledge even more and attracts their interest to studying the geometric material.

The whole store of knowledge acquired in the first and second grades will serve as a basis for studying geometric material in the third and fourth grades. In studying geometric material in the lower grades one must consider:

First: which geometric concepts should be presented to the pupils at that stage of instruction.

Second: which geometric concepts are least clear to the child at that time.

Finally: which geometric concepts are accessible to the child.

From this information, one may also propose an order of studying geometric material. At first the children should be presented an accurate concept of a specific geometric form. The child should also clearly associate the name with the form. Then the child should distinguish one form from another. In our investigation we discovered that pupils of the auxiliary school inadequately distinguish geometric concepts of the ball and circle, cube and square, rectangle and parallelepiped (beam).

In the first studies in the third grade it is advisable to present the geometric concepts of ball and circle, and their differences, and of cube and square, and of rectangle and parallelepiped (beam). It is no accident that the first geometric forms we consider are the circle and the ball. Those are the most familiar geometric forms, with the fewest essential features. Thus in first becoming acquainted with these geometric forms, the pupils need not acquire many concepts from such forms. The transition from the less to the more complicated forms will be clearer to the pupils.

The primary and the elementary form of intellectual activity is comparison, in which the likenesses and differences of geometric forms are revealed through juxtaposition and examination. The lesson should be constructed so that the pupils will work with interest. An explanatory lesson on a given topic might be constructed according to the following scheme:

The first stage of the work is direct observation of a ball. The teacher points out the "ball" and asks: "What is the name of this geometric body?" The pupils answer without hesitation. Among the answers, which may be very diverse, the teacher fixes attention on the word "ball" (he will always find one or several incorrect answers). The pupils have already become acquainted with this geometric form (in the first and second grades), associating it with the name "ball" or "little ball." In most cases the pupils have been familiar with it even before entering school.

The pupils are also to be included in the second stage of the work--in conducting laboratory work. From geometric forms (placed on a table, for example) the pupils look for objects having the form of a ball; here it is necessary to allow the children to compare and contrast, to find similar attributes of the given objects.

In working with the auxiliary school pupils, the teacher reveals a phenomenon with the question: "Why? How can you prove that?" The studies become active and reach high educational goals only when pupils develop the ability to analyze, and thus develop logical thought.

Then the teacher outlines on the blackboard a cardboard circle and asks: "What is this figure called?" From among all the answers, the pupils' attention is concentrated on the word "circle." Then the pupils select circles from among geometric forms set out on the table.

After this the pupils outline cardboard circles in their notebooks, shade them and write beneath them: "This is a circle" or simpler: "Circle." Then the teacher directs the pupils' attention to the differences between a ball and a circle. The teacher then divides the blackboard into two parts. On one side he hangs up a ball and writes beneath it "ball," and on the other side he outlines a circle and writes "circle."

In the process of the recognition of geometric forms, the greatest possible number of sense organs should be involved. The process of recognition will have a definite value only when the pupils see differences in similar objects and similarities in different objects. Without this it is impossible to form geometric concepts.

The next stage of work is modelling a ball. The pupils model a ball from clay or plasticine. This aspect of the lesson produces an emotional enthusiasm in the children, and they carry out this project with great interest. Each child strives that the ball he models might be better than his comrade's. When the ball has been modelled, he can either write down "ball" on a small paper and attach his name to the ball with the aid of a match, or write the word "ball" on the surface of the ball with a pin or a needle.

For the children to have accurate ideas of the ball and the cube and distinguish them, it is necessary once again to concentrate the pupils' attention on these geometric forms. The auxiliary school pupils must comprehensively and vividly feel the forms of the ball and the circle and see the difference between them. By all means and methods the teacher should secure the pupils' understanding that the ball is a three dimensional geometric figure, and the circle is a flat figure. Then the teacher should show several objects and ask the pupils to call out which of them have the form of a ball and which have the form of a circle.

By approximately this same plan it is possible to conduct the first lessons on the topics "Cube and Square" and "Rectangle and Parallelepiped." The concept of a triangle can be put off to a separate lesson. These lessons are to give the pupils elementary concepts of the geometric form being studied.

For a more visual conception of the course in the instruction we introduce a summary of a lesson.[8]

Summary of a Geometry Lesson in the Third Grade of an Auxiliary School

Lesson Topic: "Cube and Square"

Lesson Plan:

1. Ascertaining their ideas and knowledge of a cube.
2. Ascertaining their ideas and knowledge of a square.
3. Comparison of square and cube.
4. Modelling a cube from clay.
5. Homework.

Equipment for the Lesson:

1. Various geometric figures.
2. Cubes of different sizes.
3. Squares of different sizes.
4. Children's blocks for each child.
5. Colored squares for each child.
6. Clay for modelling cubes.
7. Rulers.

Course of the Lesson:

Teacher: Take the cubes in your right hand. Take the squares in your left hand. Zhenya, pick out all the little cubes on the table.

Zhenya put all the cubes to one side.

T: What am I holding in my hand?

P: A cube.

T: And now, Raya, pick out all the squares.

Raya put the squares to one side.

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[8] The lesson was conducted in the Auxiliary School No. 30 of Moscow by N. V. Sherkasova of the Faculty of Defectology of the Lenin State Pedagogical Institute of Moscow.

T: What is this figure called?

P: A square.

T: Everybody take a square in his left hand. Count how many sides a square has.

The pupils count the sides of the square.

P: The square has four sides.

T: And now measure the sides of the square.

The pupils measure the sides of the square and convince themselves that all sides of a square are equal.

T: And how many angles does a square have?

P: A square has four angles.

T: Take the colored squares in your hand and write on the colored side "square." And now write on the other side of the square "A square has four sides." "All sides of a square are equal." "A square has four angles."

The pupils write down the properties of the square on the squares.

T: And now let us repeat what we have learned about the square.

P: There are large and small squares. A square has four sides. All sides of a square are equal. A square has four angles.

T: Take a cube in your left hand. Point out a face of the cube.

P: This is the face of the cube.

The word "face" is written down on the blackboard and repeated several times by the pupils.

T: How many faces does a cube have?

P: A cube has six faces.

T: What form does a face of a cube have?

P: A face of a cube has the form of a square.

T: And now take clay and make a cube out of clay.

With great interest the pupils modelled a cube. Each pupil wanted to make a cube better than his comrade.

T: Point out a square. Now outline a square in the air with your finger.

The pupils outline a square.

T: And now point out in the air what form a cube has.

The pupils show a cube in the air.

T: Name some objects having the form of a cube.

P: Children's blocks.

T: Name some objects similar to a square.

P: A window, a portrait.

Homework:

Teacher: Cut out from paper as many squares as are necessary to make a cube. How many squares must you cut out?

P: Six squares.

T: Why?

P: Because a cube has six faces.

In the fourth grade a lesson of the following form may be proposed:[9]

Summary of a Geometry Lesson in the Fourth Grade of an Auxiliary School

Lesson Topic: "Ball and Circle"

Equipment for the Lesson:

1. Several balls of various sizes and colors.
2. Several circles of various sizes and colors.
3. Objects having the form of a ball or a circle: a ball of thread, a globe, several coins, the face of a wall-clock.
4. A plywood blackboard of arbitrary form, a tack, a string, and a pencil.
5. One large pair of compasses and seventeen small ones (one for each pupil).

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[9] The lesson was conducted in the Auxiliary School No. 30 of Moscow by V. Smirnova, a student in the Faculty of Defectology of the Lenin State Pedagogical Institute of Moscow.

6. Clay.

Lesson Plan:

1. Ascertaining their ideas and knowledge of a ball.
2. Ascertaining their ideas and knowledge of a circle.
3. Tracing a circle in notebooks with the aid of compasses.
4. Tracing a circle with the aid of a tack and string.
5. Modelling a ball from clay.
6. Homework.

Course of the Lesson:

Teacher: What am I holding in my hand?

Pupil: A ball.

T: You have all seen baseball players; but do you know what they play with?

P: With a playing ball.

T: What form does a baseball have?

Ps: A round form.

T: And how can you show this with your hands?

The pupils show the form of a ball with their hands.

T: What is this figure called?

P: A ball.

The teacher divides the blackboard into two parts and hangs up a ball on one side in a net and writes beneath it "ball."

T: And now name some objects having the form of a ball.

P: An apple, a balloon, a tangerine, a globe.

T: What is this figure called?

P: A circle.

The teacher draws a circle on the other half of a blackboard by outlining a cardboard circle and writes below it "circle."

The circle on the blackboard is shaded in.

T: Outline a circle in the air with your hand, and show me what form a circle has.

The pupils outline a circle in the air with their fingers.

T: Pick out objects having the form of a circle.

Pupil Abakumov picks from the table, objects having the form of a circle.

T: And now name some objects which have the form of a circle.

P: A wheel, a stadium, a button, a coin, the face of a clock, a flower-bed.

The teacher tells how a gardener traces out the base of a flower-bed, and the pupils draw a circle on the plywood with the aid of tack, string and pencil.

T: But with what else can one trace a circle?

P: With a pencil, with a cylinder, with a compass.

With the aid of the large compass the teacher shows how a circle is to be drawn.

T: And now trace a circle with the aid of compasses on sheets of paper.

The pupils trace out circles and color them with pencils; beneath each they write "circle."

T: What is this figure called?

P: A ball.

T: And now let us model a ball from clay.

With great interest the pupils began to model balls. Most balls were of the correct form. On the blackboard they then write down the names of objects having the form of a ball or of a circle (Figure 11).

Ball

Circle

Figure 11

Form of a Ball:

1) ball of thread
2) globe
3) watermelon
4) playing ball

Form of a Circle:

1) round mirror
2) coin
3) glass of a clock
4) gramophone record

Homework:

Teacher: With the aid of a coin trace a circle and shade it.

Lessons conducted according to such a plan are of great educational, correctional and practical value. The lessons proceed with much enthusiasm. The pupils enter into such a lesson with heightened interest. By various types of work the pupils get to know the geometric forms: ball and circle, cube and square, parallelepiped and rectangle. The children's attention is directed to solving one question from different points of view with maximal utilization of diverse methods.

At the end of the school year lessons of this type can be conducted in the second grade as well. A lesson on the topic "Ball and Circle" was conducted according to such a plan in the second grade of Ul'yanovsk Auxiliary School No. 39 by teacher V. N. Blagosklonova. After a series of such lessons had been conducted and the children had begun to distinguish such concepts as ball and circle, cube and square, parallelepiped (beam) and rectangle, the work was to continue in the direction of strengthening these concepts by having the children draw geometric figures with the aid of ruler, compass, and set square.

The sequence which should be observed in drawing geometric figures is the following: a) teach pupils how to draw a figure by outlining a pattern, b) teach pupils how to draw a square and rectangle of arbitrary size and of a given size by using a ruler on squared paper, c) teach pupils how to draw a square and rectangle of arbitrary size and of a given size using a ruler and a set square, but not on squared paper.

Using a set square to draw geometric figures is an important method in the instruction of the auxiliary school pupils. But insufficient attention is given to this tool in geometry lessons. In the individual investigation, not one pupil used the set square in drawing a square and rectangle. The pupils draw a right angle by sight, even though each child had a set square in front of him. The pupils of the public school stated that they had not been taught to draw a right angle with a set square.

The teacher's problem is not only to teach the pupils to draw a square or a rectangle with a set square at her direction, but also to teach the child to use this instrument independently in everyday work in drawing geometric figures. The problem is primarily to get the pupil to think about the work being done. Every advance, no matter how insignificant, develops the mentally deficient child.

One should use various methods in drawing a circle. Besides outlining with a pattern, the pupils must be taught to draw a circle with their compasses. In the fourth grade it is useful to show the drawing of a circumference with the aid of a thread, pin, and pencil.

There is another method of drawing a circumference using strips of paper. One side of the paper strip is fastened by a pin or a needle at various distances, and in the other side a hole is made for a pencil. With the aid of such an uncomplicated device, one can draw circumferences of circles of various radii. The auxiliary school pupils must be acquainted with all the above methods of drawing circumferences.

In the third and fourth grades of the auxiliary school it is possible to teach the pupils how to draw with the aid of three straight lines, and also to be able to make such lines out of matches, little sticks and the like. But what concepts can be presented to the pupils of the lower grades of the auxiliary school?

Through practical measuring and studying of the square, the fourth graders acquire the following geometric concepts:

1) A square has four angles.
2) All angles of a square are right angles.
3) A square has four sides.
4) All sides of a square are equal.

Similarly, the pupils master the following concepts about a rectangle:

1) A rectangle has four angles.
2) All angles of a rectangle are right angles.
3) A rectangle has four sides.
4) In a rectangle, the opposite sides are equal.

At this stage of instruction we still cannot define the square as a quadrilateral which has four right angles and all sides equal. We can only describe the essential features whereby this geometric figure is determined. Even here there are superfluous descriptions. For example, if a square has four sides, then it would be possible to exclude the four angles, but for the pupils of the auxiliary school it is important to emphasize both of these; for, by reason of his intellectual inferiority, the pupil of the auxiliary school, having mastered the proposition that a square has four sides, cannot answer the question "How many angles does a square have?" if he has not counted them beforehand.

In our investigation we discovered that in the fourth grade the pupils have already determined geometric figures according to individual essential features. In distinguishing a square or rectangle, different pupils focused on different features. We can conclude that the geometric concepts indicated above are completely comprehensible to fourth graders, and it would be a mistake not to use the abilities of the auxiliary school pupils.

In determining the essential features of the square, of the rectangle, and of other figures in the process of laboratory work, the pupils themselves should examine a series of squares or rectangles prepared by the teacher beforehand, for which it is necessary to count the number of sides and the number of angles, to measure the length of a side, and to measure the angles with the aid of a drafting set square. Each pupil does this independently. As a result of their work the pupils come to a definite conclusion, which is then written down on the blackboard. Then the pupils trace out a square or rectangle of given dimensions in their notebooks and write down a conclusion which they obtained from studying the sides and angles of the square or of the rectangle. Further, one should show the pupils the similarities and differences between the rectangle and the square. The pupils will acquire more accurate concepts of a geometric form only when they can find diversity in similarity and similarity in diversity.

In our investigation there were frequent cases in which the pupils did not distinguish the square and the rectangle, especially when these figures were near in form. In this case only by measuring was it possible to conclude that one form was a rectangle and the other a square. In the process of such work we not only widen the child's knowledge, but we also develop his thought.

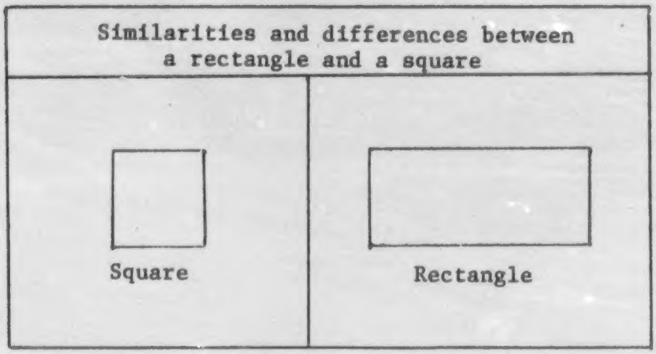
For this purpose one can take a rectangle similar to a square in dimensions and ask the pupils to name it. Most answers will be "a square," since the children relate a figure close to a square to a square. Then one should ask them to verify their assertion. How can one verify it? Only by measuring the sides. The pupil thereby convinces himself that this is not a square, but a rectangle. Further, one should give the pupils a series of figures to compare, and each time to note that it is not always possible to answer correctly and immediately the question: "What is this figure called?" As a result of such studies, we learn that it is necessary to teach children to relate critically to phenomena of the material world.

In those cases in which analysis and synthesis are necessary, the pupils develop their thought, since analysis and synthesis are aspects of a person's intellectual activity.

In comparing the rectangle and square, these figures can be distinguished only by measuring the sides, since all the other features are similar; hence the pupils should trace out the following table in their notebooks (Figure 12). In addition it is a good idea to copy such a table onto a large sheet of paper and to hang it up as a visual aid.

In studying the cube and parallelepiped, the children should be presented with the following concepts (Figure 13), in which one should also emphasize similarity and diversity.

At this stage the pupils acquire the new concept of "face," and in the fourth grade one can acquaint pupils with the name "parallelepiped." The pupils must be acquainted with the varieties of the parallelepiped. The faces of a parallelepiped (beam) may be either all rectangles, or four rectangles and two squares, or all squares (the cube is a special case of a parallelepiped). One can show this especially well in the process of modelling geometric bodies.



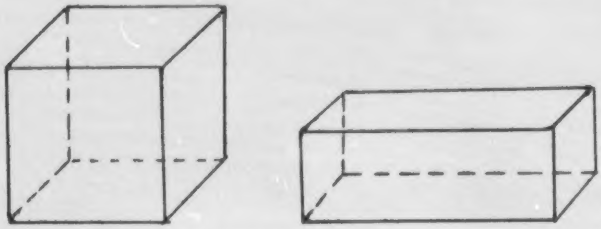
Similarities

1) A square has four angles.	1) A rectangle has four angles.
2) All angles of a square are right angles.	2) All angles of a rectangle are right angles.
3) A square has four sides.	3) A rectangle has four sides.

Differences

4) All sides of a square are equal to each other.	4) Opposite sides of a rectangle are equal.

Figure 12



Cube

Parallelepiped (beam)

Similarities

1) A cube has six faces.	1) A parallelepiped (beam) has six faces.
2) All faces of a cube have the form of a square.	2) All faces of a parallelepiped (beam) have the form of a rectangle or of a rectangle and a square.
3) A cube has 12 edges.	3) A parallelepiped has 12 edges.
4) A cube has 8 vertices.	4) A parallelepiped has 8 vertices.

Figure 13

The study of cube and parallelepiped (beam) should be conducted with obligatory modelling. When the pupil makes a model of a cube, it can then be given to another pupil to verify whether his comrade modelled a cube well, and if it must be corrected so that the cube will have the correct form. By measuring faces the pupils determine whether the faces of the modelled cube have a square form; if not, it is not a cube.

In our investigation we discovered that in the process of modelling geometric bodies, not only in the auxiliary school but also in the public school, most of the pupils modelled a cube in the form of a parallelepiped with a square base. It is not easy to reproduce a cube correctly by modelling, and in this case the pupils should develop a certain skill in carrying out the problem correctly. For variety in the work, one can propose the following check: each of two pupils sitting next to each other determines, by measuring the faces of a cube, whether his neighbor has carried out the task correctly.

To make the process of becoming familiar with geometric forms as complete as possible in the lower grades, all forms of study should be conducted with maximal use of practical projects in class, at home, on the school grounds, and in the field. For practical purposes the pupils must be able to measure the length, the width, and the height of the classroom and of a room; and they must also know how to construct geometric figures not only on paper and on the class blackboard but also on the ground. For constructing a right angle in a certain plane a T-square may be used. The pupils' knowledge of the elements of visual geometry will have value only when all forms of practical projects and all the pupils' abilities are used in studying geometric forms. Here in the lower grades the studies should be conducted systematically, in a definite sequence, and should be planned for the entire school year. Unsystematic studies, put off until the end of the school year, will not produce the effect needed.

The mentally deficient child should be taught to distinguish a geometric form, to name it correctly, to find it in his environment, and to reproduce it in a drawing or by modelling. Only comprehensive study will promote better mastery of the geometric form and correction of the defect of the central nervous sytem. Before modelling a geometric body or drawing a geometric figure, the pupil must have an idea of the shape of the given geometric form, and _then_ reproduce it.

If the child simply contemplates the square and rectangle, even naming these forms correctly, this is still insufficient for his mastery of the given form. He must be able to tell why he thinks that this is a rectangle and not a square. He must be able to say that the opposite sides of a rectangle are equal. And he must be able to measure the sides of a rectangle and say that in this figure the opposite sides are equal, and hence it is a rectangle (in the presence of the other properties of a rectangle). Only reasoning on the part of the pupil has definite pedagogical value: it trains him to think and to make a deduction. It is especially necessary to see that this occurs in the auxiliary school.

Arousing curiosity in the children is a very important aspect of instruction.

Without a system of preliminary exercises in the lower grades, the further study of geometric material and the elaboration of the fundamental practical skills will be more difficult. Preliminary studies of geometric material by auxiliary school pupils in the lower grades have both educational and correctional significance.

2. Methods of Studying Geometric Material in the Upper Grades of the Auxiliary School

From the fifth to the seventh grades, studies of geometric material are to be conducted regularly. The program stipulates that there be a required daily lesson throughout the whole school year. By this time, the pupils should have a definite system of knowledge of geometric forms. The pupils should name geometric figures and bodies correctly, should be able to distinguish them from other forms, and should also be skilled in elementary geometric drawing. Only in this case will the lessons in the study of geometric material in the upper grades be conducted successfully and on a high level.

Most pupils of the auxiliary school complete their education and then go into manufacturing; hence an elementary geometry course should be somewhat different from the studies of visual geometry in the public elementary school. The course should be systematic. One must give the pupils a complete knowledge of the geometric ideas and concepts accessible to them.

Logically and practically the study of a systematic course of elementary visual geometry in the upper grades of the auxiliary school should begin with the simplest geometric concepts.

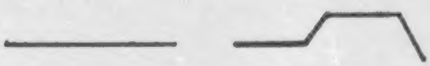
We consider it advisable to begin the study of geometric material with the point. The point is a primary concept. One should not define a point. A model of each primary concept can and should be shown. One must show how it is possible to set down a point and how a point should be designated. On the blackboard the teacher uses chalk to mark one point and says that he has put down a point; then he puts down another, and so on. Then it should be stated that it is possible to put down points in a notebook. Points will be put in the book. In order to point out something on the ground, one makes points with the aid of pegs. In practical projects the pupils make points and designate them in their notebooks, on the blackboard, and on the street.

The next concept that should be presented to the auxiliary school pupils is the concept of a line (straight, curved, and broken). The line is also a primary concept; hence one should not define a straight line. With a ruler the teacher draws a straight line on the blackboard and shows it to the pupils. Then he shows how it can be extended on both sides with a ruler.

Then the pupils draw out straight lines in their notebooks. The auxiliary school pupils should be shown repeatedly how straight lines are drawn out; as many exercises as possible are necessary on drawing out straight lines, segments, and especially segments of a particular length. To teach an auxiliary school pupil the correct use of a ruler is not very easy. We may encounter the most diverse difficulties. The teacher's problem is not only to show how a ruler is used on the blackboard, but also to show each pupil, individually, how to use a ruler.

One must perform a series of exercises with the pupils in looking for straight lines in the surroundings--in a room, on the edges of a book, on a table, on a desk, and the like, and show a straight line with a taut thread, with the aid of a thin wire, with a small stick, and the like. Also the teacher shows a bent line and a curved line.

The children draw and label these lines in their notebooks (Figure 14).



straight line broken line curved line

Figure 14

Then once more the pupil's attention is directed to the straight line and these concepts are presented: 1) straight line, 2) ray, and 3) segment of a line.

Until familiarity with these concepts is attained, it is not advisable to draw a line on the ground, or to copy a line with the aid of a card. It is advisable to conduct these practical projects on consolidating knowledge after the pupils have become acquainted with the ray and the segment.

For familiarization with the ray, a line is drawn which is bounded on one end by a point, designated by a letter (Figure 15).

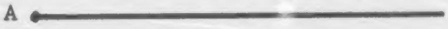


Figure 15

One must clarify that the ray may be continued as far as one likes. Then show that a line may be bounded by points from two sides, and such a line is called a segment (Figure 16).

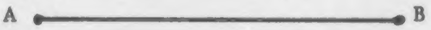


Figure 16

Here, drawings of the line, the ray, and the line segment with appropriate names under them should be made on the blackboard and in the pupils' notebooks (Figure 17). In the study of lines it is necessary

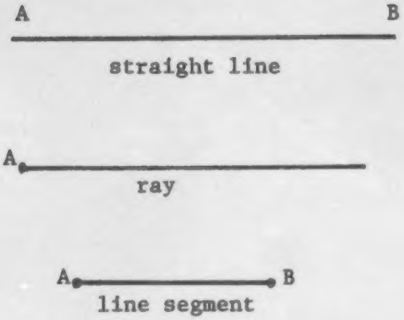


Figure 17

to show the pupils that lines, rays and segments may be drawn in different directions. The practical projects of this section are looking for straight lines in the immediate surroundings and drawing lines on the blackboard with the aid of a stretched cord as carpenters and housepainters do; it is also necessary to conduct a series of studies on drawing lines in the schoolyard or in the field.

After the pupils have been acquainted with straight line, ray and line segment, one should conduct studies in comparing segments. Here the pupils are presented with new concepts: 1) "equal segments," 2) "greater segments," 3) "lesser segment."

Sticks provide a good means of comparing segments. Then one must show the difference of segments using threads, and, after this, pass on to comparing the segments by measuring. Thus the pupils acquire a concept of the straight line; they become familiar with lines drawn on paper, in carpenter's or in housepainter's projects, and on the ground.

Further, it is necessary to acquaint the pupils with the mutual position of two lines, and with parallel lines. One should show the pupils that parallel lines do not intersect no matter how much we extend them. In a series of concrete examples show parallel straight lines: streetcar rails, a table's edges, a book's edges, and the like. In addition, it is necessary to teach the pupils to draw parallel lines with a ruler and set square.

Then the pupils become acquainted with perpendicular lines. One must teach the pupils to draw perpendicular lines with a set square.

In studying lines, special attention should be given to measuring segments. This topic is most important for the auxiliary school pupils. A necessary aspect is for the pupils to become acquainted with the units of measuring length--millimeter, centimeter, decimeter, meter, and kilometer. A segment the length of a meter, of a decimeter, of a centimeter, and of millimeter should be drawn on a paper or on the blackboard.

The most important aspect of studying geometric material is the comparison of the units of measurement--linear, square and cubic. One must clearly differentiate among these units of measurement.

The next stage of work is acquainting the pupils with angles. The angle is also not defined, but is shown visually, and the teacher says that "This is an angle." A series of exercises is conducted demonstrating angles of various objects. Then the pupils are acquainted with elements of an angle (Figure 18).

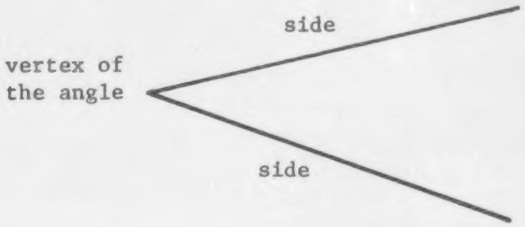


Figure 18

For a better mastery of the concept of angle it is necessary to have an angle on a hinge made of two sticks. A series of practical exercises on drawing angles is conducted with the pupils. With a moveable angle one can show an acute angle, an obtuse angle and a right angle. Further, it is also necessary to show acute, obtuse, and right angles on various objects in the environmental situation; and to provide better mastery, it is necessary that the pupils find the angles themselves.

A good tool is provided by angles cut out of colored paper. With the colored angles one can show the difference between acute, obtuse, and right angles. By comparing, the pupils should learn that an acute angle is less than a right angle, and an obtuse angle is greater than a right angle and an acute angle.

In their notebooks the pupils should draw angles and write their names beneath them (Figure 19).

Figure 19

The pupils should experience the difference between angles in cut-out constructions not only by sight but also by touch.

In the future, the pupils will often encounter right angles; hence the study of the right angle should be alotted more time than the others, and the pupils should be taught to draw a right angle with a set square. If there is no set square, then a right angle is easily made out of paper which should be folded twice for this purpose; this method must also be taught to the auxiliary school pupil.

In the sixth grade in drawing and measuring angles, a protractor is used. This section follows the intial study of the circle and circumference.

When the pupils are presented with any new concept it should be written down on the class blackboard, and the pupils should write it in their notebooks. This concept is repeated several times. At first only the good pupils repeat the name, and then the whole class repeats it. In the process of forming new concepts and for better mastery of them, the teacher must use every means at his disposal. In mastering new concepts it is necessary to return to the old ones, but to repeat them in another context so that the pupils will not be bored.

In the fifth grade one should not restrict oneself only to the study of lines and angles, but should also review the geometric figures and bodies which the pupils have studied in the third and fourth grades.

The review should consist of measuring the lengths of segments; the lengths of the sides of squares, rectangles, and triangles; the lengths of the edges of a cube and parallelepiped; the review also should consist of looking for angles in these geometric forms. In doing this, the pupils must recall the names of these geometric forms and also must reproduce them by drawing and by modelling. Otherwise the geometric material which the pupils have learned in the fourth grade will be forgotten in the fifth and sixth grades. In this case geometric forms will be visual tools and a means of better mastery of the basic geometric concepts studied as designated by the program of the fifth grade.

Sixth graders begin the study of geometric material with the circle and its circumference. The elementary concept of the circle has been mastered by the pupils in the third and fourth grades. In the sixth grade this concept is broadened significantly; the pupils become acquainted with the circle, with the circumference and the radius, with the chord and the arc, with the property of the radii of one circumference (all radii of one circumference are equal), with the diameter and its property (the diameter divides the circumference into two equal parts), and with the property of the diameters of one circumference (all diameters of one circumference are equal).

To acquaint the pupils with the circle and the circumference, one must draw a circle on the blackboard and shade it in. In their notebooks the pupils draw circumference and circle by outlining, and, in addition, the circumference is traced out with compasses, pin and thread, and strips of paper. The pupils are asked to indicate the line which has been obtained by drawing a circle; such a line is called a circumference.

The auxiliary school pupils sometimes confuse the concepts of circle and circumference. One must give this question the most serious attention. For this, one can propose the following. Draw a circle and shade it in (preferably with crayons), and write inside it "circle" (Figure 20).

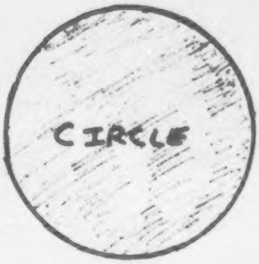


Figure 20

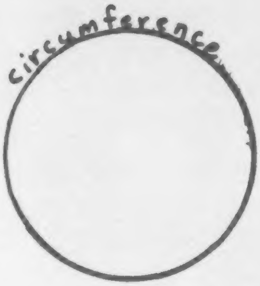


Figure 21

Then, by one of the methods indicated above, draw the circumference, and write along its border "circumference" (Figure 21). Draw a circumference, with crayon preferably, or with colored chalk, on the blackboard to give the pupils a striking image. These two drawings should be placed side by side. Then one should cut a circle out of paper; and on this model of a circle, outline the circumference with a finger. In addition one should make a circumference out of a wire. In drawing, designate the centers of the circle and the circumference. Then fold the circle along a diameter and compare the two semi-circles. The pupils do this with their circles. Having folded it in half once more, show the radius of the circle and the circumference; and on this same paper show that a circumference has any number of radii, and that all the radii of a circumference are equal. On two equal circles cut out of paper show that a circumference has any number of diameters, and that all diameters of one circumference are equal. Acquaint the pupils with other lines of the circumference and with chord and arc, of which there are arbitrarily many in each circumference.

The new concepts "diameter," "radius," "chord," and "arc," are written down at appropriate places in the circle. The pupils make similar notations in their notebooks and repeat the new words many times to learn them better (Figure 22). These lines should, for better mastery, be singled out with colored crayons. In addition, one should show that diameters, radii, and chords can be drawn in various directions. Arcs may also be drawn anywhere in a circle. For the auxiliary school pupils it is especially necessary to emphasize this fact, for

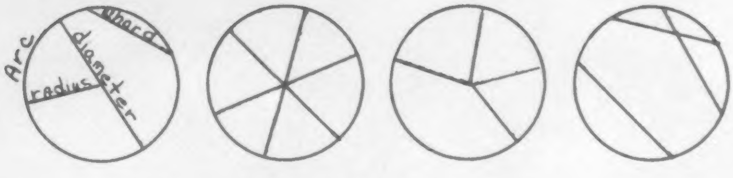


Figure 22

the pupils often draw a diameter or a radius in the horizontal position only. Such a comprehensive study of geometric concepts promotes a better learning of these concepts by the pupils of the auxiliary school.

After the section on circle and circumference, we return to angles, to measuring and constructing them with a protractor on the basis of the students' familiarity with the circle and circumference. A series of projects and exercises are conducted in measuring angles, drawing angles, and the like.

One can go on in different ways. One can begin with the study of triangle, square, or rectangle. From our point of view it is advisable to begin with the study of the triangle, because this figure is the simplest, consisting of three segments, and can be constructed with ruler and protractor, that is, on the basis of what has just been studied.

The third and fourth graders became acquainted only with the form of the triangle and, with this, with the triangle in general.

Broadening the concept of the triangle in the sixth grade, we acquaint the pupils with various types of triangles, with the altitude of a triangle, with its base, as well as with the simplest cases of constructing triangles. In addition, in studying this section the pupils become familiar with the new concept of "perimeter."

After the triangle has become familiar, it may be defined as part of the plane bounded by three straight lines. The pupils should be told that one of the sides is called the base, and a perpendicular drawn from the vertex to the base is called the altitude of the triangle (Figure 23). All this must be visually demonstrated using triangles drawn on the blackboard, cut out of paper, or made out of sticks; and the pupils should write down all these concepts in their notebooks.

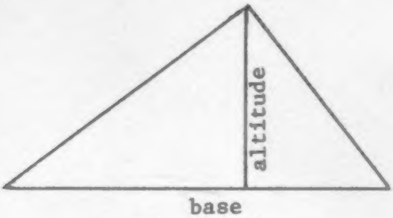


Figure 23

In connection with the study of the triangle, a series of practical projects is conducted, which consolidate a series of skills in the pupils and cultivates in them accuracy in work; in fulfilling these projects the material learned earlier is repeated. By measuring the lengths of the sides of different triangles, the pupils perceive that one group of triangles has different sides, that in another group each has two equal sides, and that in a third group all sides of each triangle are equal. The pupils conclude that triangles come in three forms with respect to their sides: scalene, isosceles, and equilateral (Figure 24). To consolidate this material one must conduct a series of practical projects to determine the different forms of triangles by measuring the lengths of the sides.

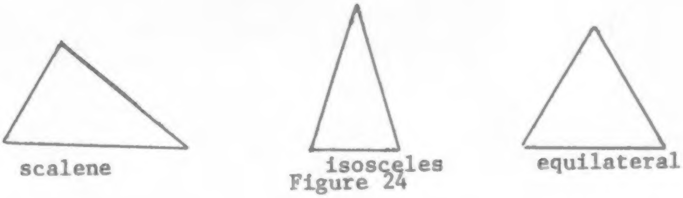
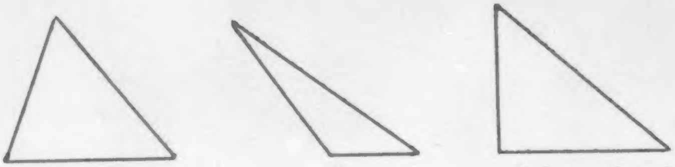


Figure 24

Then, measuring the angles of different triangles, the pupils conclude that triangles come in three forms with respect to angles: acute, obtuse, and right angles. In all cases it is necessary to conduct a series of practical projects in looking for the different forms of triangles. These forms of triangles must be written down in the notebook (Figure 25).

In this section the knowledge, abilities, and skills acquired by the pupils in becoming acquainted with the section "Angles" are reviewed and consolidated. In becoming acquainted with an obtuse



acute triangle

obtuse triangle

right triangle

Figure 25

triangle, special attention must be given to drawing the altitude. The auxiliary-school pupils can learn to draw an altitude well when the altitude passes within the triangle, but sometimes they encounter difficulties when the altitude is placed outside the triangle; that is, when it meets the extension of the base in an obtuse triangle. We observed cases in which the pupils erroneously drew the altitude in an obtuse triangle as it is reproduced in Figure 26. It is necessary to give this problem serious attention in the auxiliary school and to teach the pupils to draw an altitude in an obtuse triangle correctly.

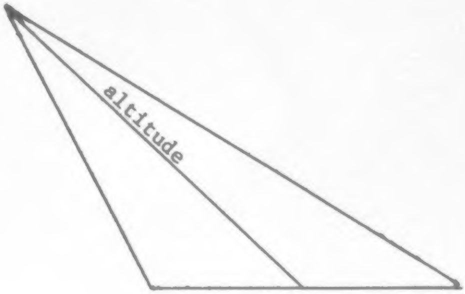


Figure 26

Using movable models one can visually show the pupils how the altitudes are placed in various triangles. In examining a right triangle it is also necessary to give special attention to drawing the altitude.

After becoming acquainted with the basic forms of triangles, the pupils are acquainted with the simplest cases of constructing triangles. One must teach the pupils to construct equilateral triangles and isosceles triangles, and also to construct triangles according to three given segments (sides), from two segments (sides) and an angle included between them, and the like.

Measuring the sides of a triangle, the pupils determine the sum of the lengths of all its sides. The sum of the lengths of the sides of a triangle is called its perimeter. This new word and new concept must be singled out very clearly. The pupils should be taught to read this word correctly. The pupils should learn the meaning of this concept accurately. In computing the areas of a square and a rectangle, the pupils often confuse "area" and "perimeter" and compute the perimeter instead of the area.

One can easily show the perimeter of a triangle composed of matchsticks or wire, by pulling the sides out into a straight line and determining the length of this line. Show that the measured perimeter of such a triangle will be equal to the perimeter computed by measuring each side separately and adding the results of the measurements.

In studying the triangle, the rigidity of the form should be noted. Show the pupils that the form of a triangle composed of matchsticks and fastened by jointpins at the angles is not subject to change. But if one fastens a square or rectangle at its angles, the forms of these figures can be changed freely, changing the square into a rhombus and the rectangle into a parallelogram. One should direct the pupils' attention to a series of objects fastened into the form of a triangle for rigidity: for example, a brace for a shelf has a triangular form, whereby greater firmness is secured for the shelf; the board on a gate is nailed obliquely; railway bridges are strengthened by triangles. Triangular forms are used very often, as the simplest form. Such wide usage of the triangle is based on its rigidity. So that the concept of a triangle's rigidity will not be formal, one must give the pupils an assignment to find in their environment different supports in the form of a triangle. Such a task has practical and correctional significance and promotes the growth of the pupils' logical thought. The pupil should find triangular supports and explain why they were made so.

In the process of studying the triangle, it is necessary to conduct a series of practical projects so that the learned concepts might have a definite correctional and educational value and the pupils might be able to use their knowledge in practice.

During the previous work, the pupils have become acquainted with lines and segments, with measuring segments, with equal segments, and with right angles. On the basis of the work performed we can deepen the study of the rectangle and the square. We can give the distinctive features of these figures, their similarities and differences, the measurement of the perimeter of these figures, and the measurement of their areas. The study of the rectangle and the square should be conducted in the following way. At first these forms are singled out from among other forms and from among objects of the environment; using awareness and perception the pupils observe these geometric forms directly.

The pupils have already been familiarized with the rectangle in the previous years of instruction. In the fourth grade, the concept of rectangle is made precise. Then the teacher passes over to the second stage of work; that is, he explains the fundamental properties of the figure being studied and directs the pupil to a correct observation and understanding of the figure being studied. The teacher directs the pupil's attention to the fact that the latter has investigated the sides and angles of the rectangle and drawn appropriate conclusions.

In the process of studying this geometric form comprehensively, the pupils establish the equality of opposite sides, the equality of all the angles, and the fact that in a rectangle all the angles are right angles. The conclusions of these propositions are arrived at inductively. The pupils are given a series of rectangles, whose angles they measure. After measuring repeatedly, the pupils conclude that all angles of a rectangle are equal. Then, measuring the sides in a series of rectangles, the pupils conclude that in a rectangle the opposite sides are equal. In this same way the pupils find out that in a square all the angles are right angles and all the sides are equal.

In the process of all this work, the pupils come to the conclusion that "a rectangle is a quadrilateral all of whose angles are right angles and whose opposite sides are equal."

Then the pupils draw a rectangle. In this a strict sequence is observed. The more exercises used and the more diverse they are, the better the concept is learned. At first the pupils trace out a rectangle from a pattern or by outlining the base of a rectangular parallelepiped; then they trace it out on squared paper without a given size and with specific dimensions; and, finally, they trace out a rectangle with ruler and set square both with and without specific given dimensions.

Tracing out a rectangle does not present great difficulties, since the pupils already know how to draw right angles and segments.

Let us consider certain aspects which must be emphasized to the auxiliary school pupils. The pupils are already acquainted with the perimeter from studying the triangle. In studying the rectangle and the square the concept of perimeter once more is recalled to the pupils and is consolidated in a series of examples and exercises, in arithmetic lessons. The pupils are also presented with the new concept of the diagonal of a rectangle and of a square.

All new concepts which the auxiliary school pupils learn in the process of instruction must be reviewed often. Repetition must be conducted by different methods so it is not boring and monotonous and so that mastery will be more thorough. Difficult words like "perimeter," "diagonal," and others must be written down on the blackboard, in notebooks; and, more often, they must be pronounced both with individual pupils and with the whole class.

A necessary condition for the best mastery of the rectangle and the square is the comparison of these figures as they are being studied. Such a method promotes better mastery of the forms being studied as well as growth of abstract thought in the auxiliary school pupil. The method of comparison must be illustrated by a figure (Figure 12).

Here the pupils must show which sides of a rectangle are equal, and which sides of a square; they must indicate the opposite sides of a rectangle. The conclusion that "the opposite sides of a rectangle are equal" simplifies computing the perimeter of the rectangle.

In the auxiliary school practice the pupils sometimes compute half the perimeter instead of the entire perimeter; hence at first in computing the perimeter one must conduct computations in the form: 5 cm + 3 cm + 5 cm + 3 cm = 16 cm. This somewhat longer notation leads the pupils to a more complete understanding of perimeter. At

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first the notation should be illustrated by a diagram (Figure 27). Later the perimeter of a figure is calculated without the diagram and only then can one pass over to the second form of notation:

5 cm x 2 + 3 cm x 2 = 16 cm.

The conclusion that "all sides of a square are equal" simplifies calculation of the perimeter of a square and facilitates the transition from notation

4 cm + 4 cm + 4 cm + 4 cm = 16 cm

to the notation

4 cm x 4 = 16 cm.

To consolidate this material it is necessary to solve a series of practical problems in determining the perimeter of the class blackboard, the area of the floor, the tabletop, and the like.

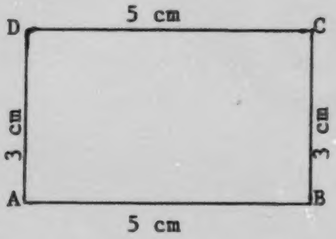

Figure 27

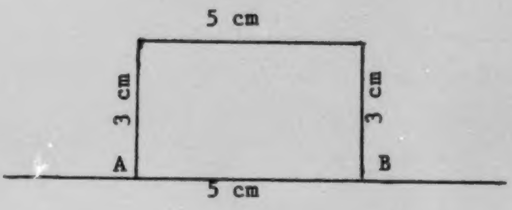

Figure 28

When the pupils have mastered the essential features of a rectangle and a square and have learned to distinguish these forms from each other, they may shift to the construction of a rectangle and a square of given dimensions with a ruler and set square. A problem is given: "Construct a rectangle with sides 5 cm and 3 cm." The construction should be done thus: a 5 cm segment, equal to one side of the rectangle, is marked off on a straight line. With the set square the right angles are then constructed at points A and B and 3 cm segments are marked off on their sides. The points obtained are joined by a line (Figure 28).

After studying the rectangle and square, the pupils go on, as has been mentioned above, to the study of square measures. We postpone the study of square measures until the next section.

Let us now examine the study of geometric material in the seventh grade. At the beginning of the school year, one must review the geometric material that has been studied in the previous years of instruction. The pupils should recall everything about the square and rectangle. One should dwell on the rectangle in somewhat more detail, note the essential features of this figure, then show a new geometric figure, the parallelogram, and compare it with the rectangle. By examining the sides and angles, we discover that the parallelogram also has four sides and that, like the rectangle, its opposite sides are equal. The parallelogram has four angles, but they are not right angles. Here lies the difference between parallelogram and the rectangle.

A valuable device with which one can show the change of a rectangle into a parallelogram is a figure composed of four little sticks fastened at the ends by joint-pins (Figure 29). In studying the parallelogram one should point out that its opposite sides are not only equal but also parallel. Then an altitude is drawn to the base. Using the parallelogram we once more consolidate the concepts of altitude and base which the pupils have studied in other figures. Now it is possible to proceed to determining the area of the parallelogram. Altitudes of the parallelogram should be drawn from different points of the base (Figure 30).

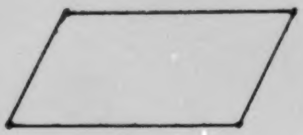

Figure 29

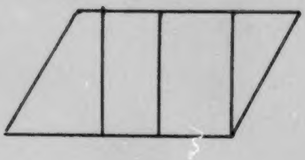

Figure 30

The word "parallelogram" is difficult to pronounce and to remember. The pupils must repeat this word often and write it down several times in their notebooks and on the class blackboard. A good device is a parallelogram cut out of paper. We shall consider determining the area of the parallelogram below, in the next section.

Seventh graders are familiar with regular polygons: hexagon, pentagon, and octagon. The pupils find these figures among others and name them. With a pattern the pupils trace out these figures in their notebooks and write their names beneath them. The pupils' attention should be directed to the fact that polygons are named from the number of their sides. If a figure has five sides, it is called a pentagon; if it has six sides, a hexagon; and if eight sides, an octagon. By themselves the pupils should find all these figures among other forms.

The section following in the pupils' text is called "Circumference and Circle; Length of the Circumference; Relation of the Length of the Circumference; Relation of the Length of the Circumference to the Diameter of the Circle; Sector." It is advisable to postpone this section until the third quarter, when the pupils study multiplication and division of decimal fractions.

We shall not dwell on the study of the form of the circle and circumference; but let us consider determining the length of the circumference. In the auxiliary school, determining the length of a circumference should be conducted only by means of direct measuring. A series of devices may be used to this end. Envelope the lateral surface of a cylinder with a wire, unwind the wire, and measure its length. The circumference of a coin may be measured by a thread or a strip of paper. Then draw a circumference on the blackboard and ask the pupils to measure its length. The problem is not beyond the abilities of the auxiliary school pupils. One must emphasize that determining the length of a circumference by winding a thread or wire about it is inconvenient, and one cannot always measure the length of a circumference by this method. Another way of measuring the length of a circumference must be found.

One should conduct supplementary work which helps the pupils determind the length of a circumference. On several circles the pupils point out the circumference and the diameter. One must remember that all diameters of the same circumference are equal. "But will the length of the circumference and the diameter be equal?" If the pupils cannot answer this question at once, they should be asked to solve this problem by measuring the length of the circumference and the length of the diameter of a coin or of other round objects. The pupils find that the length of the circumference is greater than the diameter. Then it is appropriate to pose the question: "How many times greater is the length of the circumference than the length of the diameter?"

All the measurements made should be entered in the following table. The teacher makes a table like this on the blackboard, and the children draw it in their notebooks.

Length of the Circumference	Diameter	Length of Circumference Divided by Diameter (approximately)
31.4 cm	10 cm	3.14
25.1 cm	8 cm	3.14

Then the pupils measure various circles cut out of cardboard and tell the teacher the results of their measurements. In this case it is not even required that one pupil carry out many measurements. If, in a class of thirteen to sixteen children one measures one circumference and one diameter, then thirteen to sixteen results are obtained. This is fully sufficient for the conclusion of the rule. Each pupil's results will be different, but always close to 3.14. The calculation should be carried out to the hundredths.

Then the teacher concludes that "The length of the circumference is approximately 3.14 times the diameter."

The teacher writes the conclusion down on the blackboard, and the pupils in their notebooks.

The pupils may be led to the solution of the problem in the following way:

Teacher: How many times its diameter is the length of the circumference?

Pupil: 3.14 times.

T: If one diameter is known, how can one determine the length of the circumference?

P: The diameter must be multiplied by 3.14.

The teacher formulates and writes down the rule: To determine the length of the circumference it is sufficient to multiply the length of a diameter of this circumference by 3.14. The pupils write this rule down in their notebooks and then solve a series of practical problems of this sort: "Determine the length of a circumference whose diameter is 5 cm."

Then a circumference of arbitrary radius is drawn on the blackboard. The length of this circumference is to be found. The pupils should measure the diameter and multiply the number obtained by 3.14.

To consolidate their knowledge the pupils conduct a series of practical projects.

The Study of Geometric Solids

In studying the cube and the rectangular parallelepiped the seventh graders expand their store of elementary ideas and concepts about geometric solids and also accumulate knowledge for determining their surfaces and volumes.

In studying the cube and the rectangular parallelepiped one should observe the following sequence:

a) Studying the form of the cube (face, edge, vertex); construction of a cube; determining the lateral and total surface area of a cube.

b) Studying the form of a parallelepiped (face, edge, vertex); development of a parallelepiped (face, edge, vertex); development of a parallelepiped.

c) Comparing cube and parallelepiped. One can show how to determine the total surface area of a parallelepiped, but only if the class is good; this section is very difficult for most pupils and may be omitted.

d) Determining the volume of parallelepiped and cube.

From a series of geometric bodies the pupils select a cube. For the most part they can correctly select and name this geometric body, but one may encounter pupils who do not name the body at once or will give it an erroneous name. Then on a model of a cube the pupils find the faces, edges, and vertices. Thus the pupils once more acquire concepts about the cube. By counting up the number of faces, edges, and vertices the pupils establish that:

a) a cube has six faces;

b) a cube has twelve edges;

c) a cube has eight vertices.

In measuring the face of a cube they conclude that:

d) a face of a cube has the form of a square.

Then the pupils model a cube from clay or plasticine.

By outlining the faces of a cube the pupils trace out and cut out a representation of the cube and then glue it together to form a cube. Even first graders could make a representation of the cube after a demonstration. Preparing such a representation presented no difficulties for the seventh graders.

To broaden their spatial conceptions, the pupils should know how to draw a cube on the blackboard and in notebooks in the form reproduced in Figure 31.

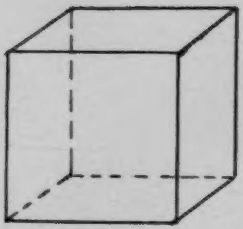

Figure 31

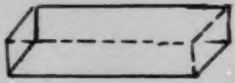

Figure 32

Similarily we examine the rectangular parallelepiped. The pupils single out a parallelepiped from among other forms. The word "parallelepiped" is difficult both to write and to pronounce; hence this word should be read several times, and written on the blackboard and in the notebooks.

After the pupils have learned the name, one should proceed to describe this geometric body. We consider it advisable to begin the study of a parallelepiped with a description, and then go on to modelling. In the process of modelling, the pupils will get a better idea of this geometric form. Then using prepared models, the basic concepts which have been acquired in the description process are consolidated.

Describing the parallelepiped proceeds roughly in the same way as describing the cube. The pupils, by counting, establish that:

1) A parallelepiped has six faces;

2) A parallelepiped has twelve edges;

3) A parallelepiped has eight vertices.

The pupils' attention should be specially directed to one final point. By investigating the faces the pupils establish that:

4) The faces of a parallelepiped may all be rectangles, or four faces may be rectangles and two faces squares, or, as a particular case, all squares.

The pupils should become familiar with parallelepipeds of the most diverse forms, and only after this proceed to preparing a model of the parallelepiped in clay.

The pupils should be taught to draw a parallelepiped in the form reproduced in Figure 32.

In studying the parallelepiped the pupils have a great choice of devices and inexhaustible possibilities in conducting practical measuring projects. Matchboxes, a pencil case, the classroom, books, cupboards, and things similar to them have the form of a parallelepiped. This form is often encountered in everyday life in their surroundings. As much time as possible should be given to the pupils' practical projects in studying the form of the parallelepiped.

An important aspect of studying the cube and parallelepiped is comparison of these geometric forms. Comparison may be conducted according to the scheme represented in Figure 13.

It is especially necessary to emphasize to the auxiliary school pupils the differences and similarities of a cube and of a parallelepiped. In this connection it is useful to conduct exercises in which one introduces to the pupils parallelepipeds close to a cube in their dimensions, and the pupils are asked to name the given geometric body. As a rule the pupils will call this geometric body a cube. Then one should ask them to prove that this is a cube. This can be done by measuring the faces of the geometric body. Such an exercise, according to the ability of the pupils, also plays a great correctional role. One must strive to show that it is necessary to consider how to prove something in all work with the auxiliary school pupils.

When the work on studying geometric forms has been completed, one can begin determining the areas of the surfaces[10] and the volumes of the bodies. We shall postpone determining the surfaces and volumes until the next section.

Next we acquaint the pupils with the cylinder; this can be done using the following plan. Introduce a cylinder and ask the pupils to name this geometric body. Some will name it correctly. Then the teacher writes down the word "cylinder" on the blackboard and reads this word several times with the pupils. From a collection the pupils pick out cylinders of various sizes. Small cylinders are distributed to the children.

The teacher describes the cylinder and points out its base and its lateral surface; the pupils point out the same on their cylinders. Then the pupils model a cylinder out of clay or plasticine, first having felt the surface of the cylinder with their hands. Then they are asked to wrap the lateral surface of the cylinder in paper and trim off the excess paper accurately. Then they outline the two bases. The pupils' attention must be directed to the figures which they have obtained in the construction. The lateral surface of a cylinder has the form of a rectangle. To determine the lateral surface area of a cylinder one must multiply the base of the cylinder by its height. The base of the lateral surface of a cylinder is equal to the length of the circumference of the base multiplied by the height of the cylinder; the length of the circumference may be computed by multiplying the diameter of the base of the cylinder by 3.14. The pupils are already familiar with computing the length of a circumference, and with the new material they only deepen their understanding. The pupils must be acquainted with devices for measuring the cylinder: with calipers, sliding calipers, and interior calipers. A series of practical projects should be conducted in measuring the diameter of a cylinder; one should also ask the pupils to name objects having the form of a cylinder, and, finally, reiterate all that the pupils have learned about the cylinder.

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[10] For brevity, in the future we shall say "determination of the lateral surfaces" instead of "determination of the areas of the lateral surface."

The pupils are acquainted with pyramid, cone, and ball by models. They find these geometric bodies among other geometric forms, model them out of clay or plasticine, and find objects having these geometric forms in the environment.[11] Modelling geometric bodies is a necessary means of instructing the pupils of the auxiliary school, since in the modelling process the pupils not only learn the studied material better, but also coordinate better the movement of their hands. They develop estimation by sight. In the process of modelling they learn a better sense of form and develop spatial imagination, which it is necessary to develop to high degree in the auxiliary school pupils.

Correctly organized studies of geometric material promote a better mastery of geometric forms and promote the general development of the personality of the auxiliary school pupil.

Pedagogical Methods in Auxiliary School Pupils' Study of Square and Cubic Measures

The study of square and cubic measure is one of the basic considerations in the instruction of auxiliary school pupils in visual geometry. The pupils master these sections well if they master the elementary concepts of area, volume, and units of measurement well. In addition, a knowledge is required of the basic concepts of the geometric form whose area or volume is to be determined.

The pupils must learn such concepts as base and altitude, length, and width, and, for geometric bodies, length, width, and height. Before proceeding to the study of square and cubic measures, all these concepts must be solidly learned by the children.

1. The Study of Square Measure

After we acquaint the pupils with the elementary concepts of square and rectangle, the next stage of work is acquainting them with the concept of area. The measurement of areas should begin with the rectangle since the pupils learn the area of a rectangle more easily. Moreover, determining the area of a square is a special case of determining the area of a rectangle.

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[11] The pupils were already familiarized with the ball at preschool age and in the lower grades. In the upper grades they only recall and deepen their understanding of what has passed before.

First we show the pupils two rectangles of different colors, where one of them is placed within the other.

Teacher: What is this figure called?

Pupil: A rectangle.

T: And what is this figure called?

P: Also a rectangle.

T: Which of these two rectangles is larger?

P: The pupils point out the larger rectangle.

T: And how can this be proven?

P: Place one on top of the other.

Then they compare the larger rectangle with another one whose area is greater still, and the pupils convince themselves that the new rectangle has an area greater than that of the given one, and that the given one's area is less than the new one's area. And, finally, equal rectangles are shown to the pupils.

By comparing the two different rectangles one can show that rectangles can be equal or different. At the first stage "larger," "smaller," or "equal," can be shown only on figures whose difference or equality can be shown by superposition. The pupils learn that figures may be equal or different.

Then we take two different rectangles, one of them with an area twice that of the other, and we ask the pupils "which figure is larger?" The pupils find the answer to this question by superimposing the figures.

"But can we determine how many times one rectangle is larger than the other?" Among the rectangles, we choose three or four of the same size as the smaller. Two pupils place the rectangles on the given ones, by superimposing, they determine that the area of one of them is twice the area of the other, for two equal rectangles fit into it.

Then the same figures are shown again, and the question is changed to "Which rectangles are equal, larger, or smaller?" The concept of the area of a square is presented in the same way.

Then the pupils are asked to determine the difference between the areas of a rectangle, and a square, when the length of the rectangle is twice the length of the side of the square, and its width half the side of the square.

"Which of these figures has the greater area?"

The pupils have difficulty in answering.

Then the teacher, by folding the rectangle/square in half and cutting along the line of the fold changes the rectangle/square into a square/rectangle of equal size. Through this demonstration the teacher shows that different figures may have equal areas.

Then the teacher, with concrete examples, shows the area of different forms: the area of the floor, of a staircase, of a corridor, of a yard, and of a garden. By superimposing, the magnitudes of the areas are compared.

Then the pupils are asked to compare the areas of two figures which cannot be compared by superimposing. A new question arises: is it possible to first measure the areas of the figures and then to compare these areas? Thus we lead the pupils to the necessity of measuring areas.

After this step it is appropriate to review the units used in measuring length. The pupils determine in their practical projects the length and width of a rectangle and a square, the perimeter of these figures, and the perimeter of a triangle. The pupils are shown a table with linear measures, and repeat the practical measuring of segments once more. They conclude that the length of a segment or of a line is measured in linear measure.

The pupils are given various units of measure: measures of weight, of angles, and of length. The pupils should understand that different measurements have their own particular units of measuring. One should show that areas cannot be measured in same units used to measure length, weight, or angle size.

Area is measured by areas. It is most convenient to measure the area by squares. One adopts square measures for measuring areas. These are special squares. Here one should review about squares in general, and then from among them, single out squares with sides of one centimeter, one decimeter, one millimeter, and so forth. Such squares are drawn in the class notebooks and below them is written sq. cm , sq. dm , and sq. mm , etc.

The pupils should be visually shown the difference between linear and square measures. In the auxiliary school it frequently happens that pupils measure area with linear measures. This is because of their inaccurate ideas of square measures.

In their notebooks the pupils should have a table like Figure 33.

Unit of measure for length	Unit of measure for area
1 mm	1 sq.mm
1 cm	1 sq. cm

Figure 33

In our investigation we frequently observed cases in which the pupils wrote down linear units when measuring areas; most serious attention should be given to this problem in the auxiliary school.

A square meter is drawn on the blackboard. The square centimeters contained in the square meter should be shaded in. The pupils should understand that 1 centimeter and 1 square centimeter are not the same thing.

A square meter should be pasted together from paper or made out of sticks. The square meters should be drawn into square decimeters, and the square decimeters into square centimeters. One should count how many square centimeters there are in one square decimeter and how many square decimeters are in one square meter. A large number of squares the size of each square unit should be cut out of cardboard. Thus, the pupils will have visual ideas of square measures.

After acquainting the pupils with square units, one should begin measuring the area of a rectangle. At first the area of a rectangle is measured by superimposing square units. Here one should conduct a series of practical projects in computing the areas of rectangular figures by superimposing. The teacher measures the areas of figures beforehand so that the square units fit exactly on the surface of the figure being measured. The area of a table, a desk, a book, and the like are measured; one should also measure a large rectangle cut out of paper. Decimeter squares are put next to each other, and then their number is counted and written down as: the area of the table = 80 sq. dm. The inconvenience of this method of measuring is pointed out to the pupils, and it is proposed that they determine the area by another method.

A rectangle is traced out. By covering it with square decimeters, one divides it into a series of squares. These are then counted. In this case the teacher points out to the pupils that it is possible to measure the area of a rectangle using only one square unit of measurement.

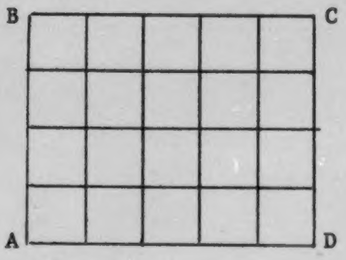

Figure 34

After the rectangle has been divided up into squares (Figure 34), its area is determined by directly counting the squares obtained; then the teacher states that it is possible to determine the area of a rectangle in another way, and the following dialogue is conducted.

Teacher: How many square decimeters fit into the lower row?

Pupil: 5 sq dm. (This is determined by counting directly).

T: How many fit into the second row?

P: 5 sq. dm. And so on.

T: How many such rows of 5 sq. dm each are there?

P: Four rows.

T: Then how many squares are there altogether?

P: One must take 5 sq. dm 4 times.

T: This can be written as:

P: Area of the rectangle = 5 square decimeters x 4 = 20 square decimeters.

One may conduct a lesson on determining the area of the rectangle according to the following plan:

Summary of a Geometry Lesson in the Sixth Grade of an Auxiliary School[12]

Topic: "Measuring the Area of a Rectangle"

Lesson Plan:

1. Reviewing the material on the rectangle.
2. Familiarization with the units of measuring area.
3. Drawing a square decimeter, a square centimeter, and a rectangle in notebooks.
4. Measuring the area of a rectangle by covering one with square decimeters on the blackboard and with square centimeters in the notebooks.
5. Measuring the area of a rectangle by measuring the base and the altitude.
6. Formulation and notation of the conclusion.
7. Problem solving.
8. Homework

Equipment for the Lesson:

Cardboard squares the size of a square meter, square decimeters, square centimeters; rulers, set squares, and paper.

Course of the Lesson:

Teacher: What is the figure called?

Pupil: A rectangle.

T: What kind of figure is a rectangle?

P: A rectangle is a quadrilateral in which the opposite sides are equal and the angles are right angles.

T: Point out the base and the altitude.

The pupil points out the base and the altitude.

P: One of the sides is taken as the base, and the side perpendicular to it is taken as the altitude.

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[12] The lesson was conducted in the Ul'yanovsk Auxiliary School No. 39 by teacher M. A. Golubchikova.

T: Today we will measure the area of a rectangle. To measure the area of a rectangle one must know how many units of measurement fit into this area. With what units of measurement can one measure the area of a rectangle? (The pupils are already familiar with the units of measuring areas.)

P: With square meters, square centimeters, and square decimeters.

The teacher shows the units of measuring areas, cut out of paper.

T: Show a square meter, a square decimeter, and a square centimeter.

A girl indicates all the units of measurement correctly.

T: Why do you think this is a square meter?

P: (measures a side of the square): Because this is a square with a side of one meter.

Another girl measures the side of a square decimeter.

T: Why is this figure called a square decimeter?

P: Because a side of this square equals 10 centimeters or 1 decimeter.

The pupils measure the sides of square decimeters and centimeters which are on their desks and point out the square decimeters and the square centimeters.

T: Once again name the units of measuring area.

Ps: Square kilometer, square meter, square decimeter, square centimeter, and square millimeter.

T: Examine how square measures are designated. Let us write them down on the blackboard: sq. km ; sq. m ; sq. dm ; sq. cm ; sq. mm.

A pupil reads these square units of measurement.

T: And now measure the side of a small square. What is the length of a side of this square?

Ps: One centimeter.

T: What is such a square called?

Ps: A square centimeter.

T: (calling one of the pupils to the blackboard): Write down the notation for a square decimeter. Pay attention to how a square decimeter is written. In your notebooks draw square decimeters and square centimeters. And now everybody write down in your notebooks the topic of today's lesson.

The pupils write the topic of the lesson in their notebooks.

On the blackboard a rectangular sheet of paper is fastened on which a rectangle is drawn (Figure 35).

Figure 35

T: We must measure the area of rectangle ABCD. With what kind of units can we measure this rectangle?

Ps: With the square decimeter.

T: To measure the area of the rectangle one must find out how many times a square decimeter fits in this area.

The teacher shows how the area to be measured is covered by square decimeters, and two square decimeters are outlined. Then one pupil is called up; she sections the area of the rectangle by superimposing and outlining the square decimeter (Figure 36).

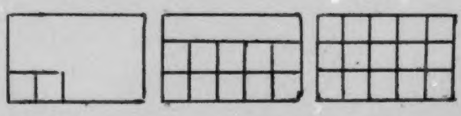


Figure 36 .

T: And now count and tell us how many square decimeters fit into this rectangle.

Ps: 15 square decimeters were fitted into this rectangle.

T: What is the area of this rectangle?

Ps: The area of the rectangle equals 15 square decimeters. This may be written down as: The area of rectangle ABCD = 15 square decimeters.

T: With what measures can one measure the area of the floor?

Ps: With square meters.

T: And how shall we measure it?

Ps: We lay out square measures on the floor.

T: But is it convenient to measure it thus?

Ps: No, it is not convenient. The tables and desks will hinder us.

T: And if one had to measure the area of a forest or a field, with what units could one measure them?

Ps: With square kilometers.

T: But would it be convenient to measure by covering?

Ps: No.

T: And now draw in your notebooks a rectangle with sides of 5 cm and 3 cm. What is the length of the base of this rectangle?

Ps: The base of the rectangle equals 5 cm.

T: How many square centimeters fit on the base?

Ps: 5 square centimeters.

T: What is the height of the rectangle?

Ps: The height equals 3 centimeters.

T: How many times does 5 square centimeters fit into this rectangle?

Ps: Three times.

T: How can this be written down?

Ps: 5 square centimeters x 3 = 15 square centimeters.

T: What is the area of the rectangle?

Ps: 15 square centimeters.

T: How can one write this down?

Ps: Area of the rectangle = 5 sq. cm x 3 = 15 sq. cm.

T: How can one measure the area of the rectangle in another way?

Ps: One should measure the base and the altitude (the length and the width) and multiply the numbers obtained.

T: How can one measure the area of the floor now?

Ps: One should measure the length and the width of the floor and multiply the numbers obtained.

The teacher calls on a pupil to measure the length and the width of the floor and compute its area.

Let us solve this problem: Determine the area of a rectangle whose base is 15 centimeters and whose height is 3 centimeters.

Ps: The area of the rectangle = 15 sq. cm x 3 = 45 sq. cm.

T: How can one determine the area of a rectangle?

Ps: To determine the area of a rectangle one should
measure the base and the height (the length or
the width) and multiply the numbers obtained.

The pupils write down the conclusion in the notebooks.

Homework:

Measure the length and width of a room and determine its area.

After the pupils are given an initial acquaintance with the measuring of area, one assigns a series of practical projects in computing the areas of rectangles by covering the figure with square units of measurement thereby dividing the given figure into square units. Then the areas of rectangular figures are computed only by dividing them up into square units. But the teacher should measure the length and width of the rectangular forms beforehand to make sure it is possible to measure the areas by division into squares.

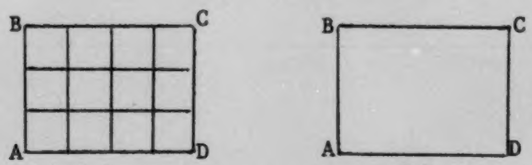
In studying square measures in the auxiliary school, this stage of work is very important although it takes much time. In determining the areas of rectangles and squares the auxiliary school pupils often mechanically multiply the length by the width and do not understand the sense of determining areas; they will understand it only when they discover, through practical projects, that to measure an area means to find out how many square units fit into the given area.

If we omit this stage of determining areas in the auxiliary school, the later projects on the topic of square measures will bear a formal character, and will not have correctional and educational significance. One should also measure areas in the field, determining the area of a rectangular lot. Instead of dividing the lot into squares, one should merely drive in pegs and count the squares formed with each peg serving as a vertex.

Only after this work is it possible to lead the pupils to the discovery that there is another, easier method for determining the areas of figures. Before the pupils are shown this method, many examples of the inconvenience of the first and second methods should be pointed out. For this purpose one should propose to measure the area of a rectangle or of an object having a rectangular form on which a whole number of units of measurement do not fit, or propose to measure the area of the classroom. The tables, desks, and cupboards will be obstacles which, to a significant extent, will make it difficult to determine the area of the classroom.

Emphasize to the pupils that it is sometimes even impossible to measure an area by this method. For example, one cannot measure large areas of ground by the method of covering. The pupils cannot lay down and pick up such units of measurement as the square kilometer, hectare, or acre. In measuring, one encounters villages, forests, lakes, or swamps within or on the border of a lot, so that one cannot determine the areas of rectangular lots by the device of covering.

In the class the teacher then shows on the blackboard how to determine the area of a rectangle by the usual method. The blackboard is divided into two parts and on both small rectangles of equal size are traced out and the areas of these figures determined by the two methods (Figure 37):



(a) (b)

Figure 37

(a) This rectangle is divided into squares and, on the basis of reasoning, the area of the rectangle is written down as:

"The area of the rectangle equals 4 sq. dm x 3 dm = 12 sq. dm."

(b) This rectangle is not divided into squares, and the line of reasoning is:

T: What is the length of the rectangle?

P: 4 decimeters.

T: How many decimeters fit into the length?

P: 4 square decimeters.

T: What is the height of the rectangle?

P: 3 decimeters.

T: How many rows 4 square decimeters long will fit into the rectangle?

P: Three rows.

T: How many square decimeters fit into the whole area?

P: 4 square decimeters x 3 = 12 square decimeters.

T: What did we do to determine the area of the rectangle?

P: We measured the length and the width and multiplied the numbers obtained. The area of the rectangle = 4 sq. dm x 3 = 12 sq. dm.

T: What answer did we get in the first case?

P: 12 square decimeters.

T: And what answer did we get in the second case?

P: 12 square decimeters.

After this another pair of rectangles should be measured in the same way.

In determining the area of a rectangle in the auxiliary school such a parallel exercise must be conducted. Then the pupils conduct a series of exercises in determining the areas of rectangular figures by measuring the width and the length (the base and the height).

Now the pupils can compare the areas of two figures by measuring, and can solve the problem they could not solve before.

In the auxiliary school we shall not teach the method of computing areas in the form presented in Figure 38.

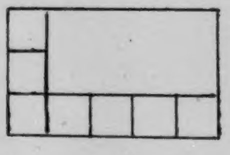


Figure 38

Such a method will create an incorrect geometric image, and in determining the area of a rectangle, the pupils may compute the perimeter in square units, as happened in our investigation. The incorrectness of this method is pointed out by P. A. Kompaniits:

> This method is completely inadmissible, since it gives an incorrect geometric image. The above-mentioned answer of a pupil "The length should be multiplied by the width," giving, as an explanation, "For one should find out how much they are together" is explained by the use of this method in the instruction [6:55].

We consider that in the auxiliary school only one form of notation should be used in computing the area of a rectangle, and this is:

Area of Rectangle = 4 square decimeters x 3 = 12 square decimeters.

This form will be quite sufficient for the pupils' solution of all possible problems in computing areas.

There is another form of notation, 4 dm x 3 dm = 12 sq. dm , which is unacceptable in the auxiliary school. The meaning of this notation is not understandable to mentally deficient children. They cannot understand why, when multiplying linear measures by linear measures, they obtain square units of measurement. This form of notation more often leads the pupils into errors, and the pupils write down the solution of a problem in the most diverse ways:

a) 4 dm x 3 = 12 dm ;
b) 4 dm x 3 dm. = 12 dm ;
3) 4 dm x 3 = 12 sq. dm.

We observed this form of notation in our investigation. In this form the solution of the problem loses its meaning.

When the pupils use only one form of notation, without excess units, and when the final result must be in square units, they always solve the given problems correctly.

When the computation of the areas of rectangles has been learned well, one must pass on to solving practical problems; here one should conduct a series of practical projects in calculating areas, not only from given dimensions, but, for the most part, also from given forms. In the auxiliary school, special attention must be given to computing the areas of rectangles when their forms are given. In our investigation we discovered that the most difficult problems are problems in computing the areas of given forms. Here it is necessary first to determine the dimensions, and then the area. Then one should go on to solving problems in computing areas.

Computing the area of a square is done in the same way as computing the area of a rectangle. Here one must not neglect the work conducted in calculating the area of a rectangle, since it is important for the pupils to understand the meaning of computing the areas not only of rectangular forms but also of square ones. In our investigations we observed that in calculating the area of a square the pupils encountered more difficulties than in determining the area of a rectangle. In determining the area of a square one should point out that the length and height of a square are equal; and that to determine the area of a square one should multiply the base of a square by itself.

When working on practical projects, the pupils had to be shown the measures of ground areas: are and hectare. To acquaint the pupils with the are and the hectare one should conduct a lesson in the field (outside the city) or somewhere in a park. The pupils become visually acquainted with a square whose side is 10 meters long. Such a square is called an "are" or, in general usage, a sotka (hundredth part of a hectare). In constructing a square on the ground with a side of 100 m , the pupils come to know the hectare.

The pupils will have a complete and accurate idea of square measures and the measurements of areas only when they have become acquainted with the various unit measures and the various methods of measuring them. In calculating the areas of a rectangle and of a square, one should give special attention to the difference between computing the area and computing the perimeter of a given figure. The auxiliary school pupils often do not distinguish accurately enough between the concept of determining the area and that of determining the perimeter. It is necessary to show that these are not the same.

We can indicate the following basic stages in studying square measures and measuring areas:

a. giving the pupils an accurate concept of area as the space contained within a figure;

b. giving the pupils visual acquaintance with square measures: square millimeters; square centimeters; square meters; are and hectare;

c. measuring the area of a rectangle (square) by completely covering it with square units;

d) measuring the area of a rectangle (square) by dividing it into square units: by dividing, with a unit square as a successive cover; and by outlining and by dividing into square units with a ruler; and by constructing squares within the figure;

e) determining the area of a rectangle (square) by measuring length and width (base and height) and multiplying the numbers obtained;

f) consolidating the conclusion obtained in determining the area of a rectangle ("To determine the area of a rectangle one should measure the length and width and multiply the numbers obtained") by conducting a series of practical projects in computing areas of rectangles and of squares.

In the auxiliary school one should give attention to determining the area of a right triangle and that of a parallelogram. The area of a right triangle is determined as half the area of the rectangle having base and height equal to those of the triangle.

We should like to note one more exercise which is not included in the auxiliary school curriculum--showing the pupils that every triangle can be turned into two right triangles whose areas can be determined.

The area of a parallelogram is determined thus: The parallelogram is changed into a rectangle of equal size, and the area of the rectangle is determined. Thus the area of a parallelogram is defined as the product of the base and the height.

The ability to calculate the area of a right triangle and of a parallelogram is consolidated by a series of practical exercises. Determining the surface area of a cube reduces to determining the areas of squares. Each face of the cube is first divided into square units of measurement and the areas of the separate faces are computed; then the full or lateral surface area of the cube is computed.

The surface area of a cube can be determined by measuring the edges of the faces. Since all faces of a cube are equal, it is sufficient to determine the area of only one face and to multiply by six to determine the complete surface area and by four to determine the lateral surface area of the cube. By this same method one determines the complete and lateral surface areas of a rectangular parallelepiped, in which it is necessary to consider that the opposite faces of a parallelepiped are equal. In all cases, when determining the areas of figures, one must give very serious attention to the form of notation.

The practical use of square measures must be conducted in the school yard and in the field so that one can not only consolidate this section but also review linear measures and recall the construction of geometric figures in the vicinity.

2. The Study of Cubic Measures

Before showing them how to measure volumes it is necessary to present to the pupils the concept of the volume of a body. For this purpose the preliminary work described below is conducted. The pupils already have elementary ideas and concepts of geometric bodies. One must recall to them that every body occupies a position in space. Different bodies occupy different positions in space. If a geometric body is empty inside, then it has a definite capacity.

To visually show the capacities of bodies, one must display several different tin or cardboard boxes with an open face. For this purpose it is good to use a set of children's blocks, each of which fits inside the one of the next larger size. One can point out which cube is greater (i.e., occupies more space); the part of space which a given geometric body occupies is called the volume of this body. Or, to put it another way, the capacity of a given geometric body is called its volume.

With the different cubes one can show which body has the greater volume. At first this can be shown by putting one geometric body inside another. If one cube is greater than a second, the volume of the first will be greater than the volume of the second. Then one can fill the cubes and parallelepipeds with any substance. For example, one can pour water in or fill them with sand. In this way one can show the pupils that different bodies have different volumes, and that there are geometric bodies which have equal volumes.

When the pupils have been visually acquainted with the volumes of bodies and with their capacities, and have a visual idea of the volume of a body as its capacity, one can learn the units by which one measures volumes. Before naming the units for measuring volumes, one should recall the linear and square units of measurement. The pupils name and draw on a blackboard the linear and square units of measurement.

In studying the cube one should consider cubes whose edges are one centimeter, one decimeter, one meter, and one millimeter. These will be special cubes.

After the pupils have named the linear and square units, the teacher himself explains that for measuring volumes corresponding measures are used: one cubic meter, one cubic centimeter, 1 cubic decimeter, and 1 cubic millimeter. One may ask the pupils beforehand whether the volumes of a cube and of a parallelepiped can be measured in linear or square units. From the very beginning the pupils should learn that the volumes of bodies are measured with cubic units of measurement. These units of measurement must be shown to the pupils visually.

One cubic centimeter and one cubic decimeter are in the arithmetic box* (or it is easy to construct them out of cardboard). A cubic meter can be made from plywood or from six pieces of cardboard, each the size of a square meter, with sticks attached around them. Fasten these together with loops of wire.

The pupils must be given a visual idea of the cubic meter. It is somewhat more difficult to make a cubic millimeter. An approximate idea of the cubic millimeter may be given to the pupils by cutting one out of plasticine. To give the pupils a more accurate idea of the cubic units of measurement, one should construct a table of measures on the wall having a form like Figure 39.

Such a table may be made by the pupils themselves. A square millimeter, square centimeter and square decimeter should be colored in, but a cubic millimeter, cubic centimeter, and cubic centimeter should be modelled out of wood or out of clay or plasticine and attached to the cardboard by wire or a thin cord. So that the pupils might have a realistic idea of the units of measuring length, area, and volume, these units should be represented in the table at their natural size.

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*Box of manipulative materials (Ed.).

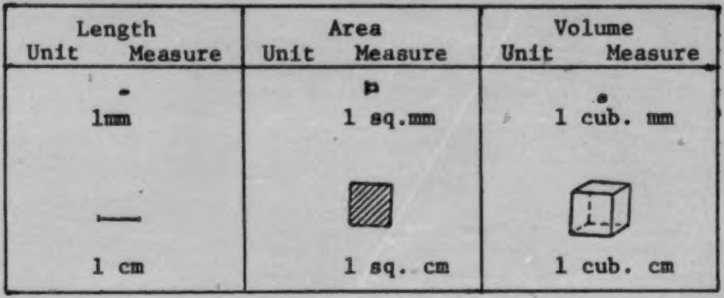

Figure 39

Such visual representation of the units of measurement promotes more accurate mastery of square and cubic measures.

The pupils should learn clearly that one can measure volumes only with cubic units. After the pupils have been acquainted with the units of measuring volumes, one should begin measuring the volume of a rectangular parallelepiped. At first the volume of a parallelepiped is measured by a complete insertion of blocks the size of the units of measurement into the given geometric form. For this purpose it is necessary to have the pupils build, in the workshop, a plywood or a wooden (be sure to use thin boards) box of these dimensions: length, 4 decimeters; width, 3 decimeters; and height, 2 decimeters. Or they can make a parallelepiped of dimensions 3 decimeters, 2 decimeters, and 4 decimeters with the front face open. They should also cut 24 cubes; each should be the size of a cubic decimeter. These can be painted any color.

The volume of the parallelepiped is determined thus. At first decimeter cubes are set out along the length.

Teacher: How many cubic decimeters fit along the length?

Pupil: Four cubic decimeters.

Then three such rows with 4 cubic decimeters in each are set out.

T: And how many such rows fit onto the bottom of the box?

P: Three rows.

T: How many cubic decimeters fit into each row?

P: 4 cubic decimeters each.

T: How many cubic decimeters fit into the first layer?

P: One must multiply 4 cubic decimeters by 3.

T: How can this be written down?

P: 4 cubic decimeters x 3 = 12 cubic decimeters.

Then another such layer is set out.

T: And how many such layers are there?

P: Two.

T: How many cubic decimeters fit into each layer?

P: 12 cubic decimeters each.

T: And how many cubic decimeters fit into the rectangular parallelepiped?

P: It must be 12 cubic decimeters x 2 = 24 cubic decimeters.

T: How else can this be written down?

P: 4 cubic decimeters x 3 x 2 = 24 cubic decimeters.

T: Because 4 cubic decimeters fit into each row on the bottom of the box, there are three such rows which fill up the bottom of the box, and there are two such layers. Then what is the volume of the parallelepiped?

P: The volume of the parallelepiped equals 4 cubic decimeters x 3 x 2 = 24 cubic decimeters.

The pupils can measure the volumes of small boxes by filling them with cubic units of measurement. We propose that one determine the volume of a parallelepiped or a cube only by filling it completely with cubes the size of the units of measurement, and then begin determining the volume of a cube by directly measuring the length, width, and height and multiplying these numbers.

After acquainting the pupils with finding the volume of a parallelepiped or a cube by filling a model of it with cubic units of measurement, one should note that such a method of determining the volume of a cube or parallelepiped cannot always be used. For example, to determine the volume of a classroom by this method does not seem possible, since one cannot empty the classroom of furniture and make cubic meters especially for this purpose; and if a pupil has to measure a solid parallelepiped rather than a hollow one, then its volume cannot be measured at all by this method. There are simpler, more convenient methods. To lead the pupils to such a method of determining volumes it is necessary to conduct a conversation of this nature with them:

Teacher: How many cubic decimeters fit along the length of the box?

Pupil: 4 cubic decimeters.

T: And what is the length of the parallelepiped?

P: 4 decimeters.

T: How many such rows of 4 cubic decimeters each will fit onto the bottom of the box?

P: Three rows.

T: What is the width of the cube?

P: 3 decimeters.

T: How many cubic decimeters are there in all?

P: 4 cubic decimeters x 3 = 12 cubic decimeters.

T: How many such layers of 12 cubic decimeters each are there?

P: Two layers.

T: What is the height of the parallelepiped?

P: 2 decimeters.

T: How many cubic decimeters fit into the box altogether?

P: 12 cubic decimeters x 2 = 24 cubic decimeters.

T: How else can this be written down?

P: 4 cubic decimeters x 3 x 2 = 24 cubic decimeters.

T: What is the length of the parallelepiped?

P: 4 decimeters.

T: What is the width of the parallelepiped?

P: 3 decimeters.

T: What is the height of the parallelepiped?

P: 2 decimeters.

T: What should be done to find out the volume of the parallelepiped?

P: It is sufficient to find out the length, width, and height of the parallelepiped and to multiply the numbers obtained. This can be written as:

4 cubic decimeters x 3 x 2 = 24 cubic decimeters.

The pupils formulate and write down the rule for calculating the volume of a parallelepiped: "To calculate the volume of a rectangular parallelepiped it is sufficient to measure the length, the width, and the height and to multiply the numbers obtained." After this, this rule is implemented in a series of problems. Then, when the pupils have solidly mastered determining the volume of a rectangular parallelepiped, one should pass on to the volume of a cube.

In determining the volume of a cube, one should also measure its volume by filling it completely and by directly measuring the length, width, and height. In determining the volume of a cube one should point out that in a cube the length, width, and height have the same measurement; hence to determine the volume of a cube it suffices to measure only the length of an edge and to repeat the number obtained as a multiplier three times. One can go on to this only after the pupils have thoroughly understood the measuring of the volume of a cube. Otherwise the pupils may multiply the length of an edge by three.

One may conduct a lesson in determining the volume of a parallelepiped according to the following plan:[13]

Summary of a Geometry Lesson in the Seventh Grade of an Auxiliary School

Topic for the Lesson: The volume of a parallelepiped

Equipment for the Lesson:

A model of a parallelepiped without a front face, and twelve cubic decimeters, cubic meters, and cubic centimeters.

Lesson Plan:

1. Review of the previous material on the parallelepiped.
2. Explanation of the new material.
3. Consolidation of the conclusion by solving problems.
4. Homework.

Course of the Lesson:

Teacher:	Point out a parallelepiped and tell all that you know about it.
Pupils:	A parallelepiped has six faces, eight vertices, and twelve edges. Each face of a parallelepiped has the form of a rectangle.

---

[13] The lesson was conducted in the seventh grade of the Ul'yanovsk Auxiliary School by Z. M. Kazanskaya.

Pupil Danilova pointed out a parallelepiped, indicated its faces, edges, and vertices, and told everything about a parallelepiped.

T: Point out a cube and tell all that you know about it.

[Note: The model of the parallelepiped and the cubic decimeters were prepared by the seventh graders in the school workshop.]

Ps: A cube has six faces, eight vertices, and twelve edges. An edge of a cube has a square form.

A pupil picked out a cube from among geometric bodies and told everything about the cube.

T: What is similar about a cube and a parallelepiped?

Ps: Both the cube and the parallelepiped have six faces, eight vertices, and twelve edges.

T: What distinguishes a parallelepiped from a cube?

Ps: In a cube all the faces are equal, but in a parallelepiped the opposite faces are equal, and they are rectangles.

The teacher repeats the similarity and difference between a cube and a parallelepiped.

T: Now let us determine the volume of the parallelepiped. What units of measuring length do you know?

Ps: Meters, centimeters, decimeters, kilometers, and millimeters.

T: What units of measuring area do you know?

Ps: Square meters, square decimeters, square centimeters, square millimeters, square kilometers, ares, and hectares.

T: And can one use linear or square measures to measure volumes?

Ps: No. Volumes are measured in cubic measures.

T: With what measures can one measure the length of the classroom?

Ps: With a meter.

Pupil Seliverstova took a meter from the table and measured the length of the classroom. She wrote down on the blackboard:

length of the classroom = 5 meters.

T: And with what measures is the area of the classroom measured?

Ps: With square measures.

T: How can one measure the area of the classroom?

Ps: One should measure the length and the width and multiply the numbers obtained. Or it could be measured by covering it with square meters.

T: When we measure the length of the classroom what else should be measured?

Ps: The width of the class.

Seliverstova measured the width of the classroom and wrote:

Width of the class = 4 meters.

T: What is the area of the classroom?

Ps: The area of the classroom = 5 square meters x 4 = 20 square meters.

T: And now let us measure the volume of a parallelepiped. With what units of measurement is volume measured?

Ps: With cubic units.

T: We shall measure the volume of this parallelepiped.

The teacher shows the box with the front face open. We shall measure it by filling it with cubic units (Figure 40).

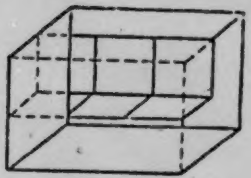

Figure 40

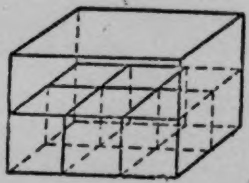

Figure 41

Along the length of the parallelepiped, they pack 3 cubic decimeters.

T: What is the length of the parallelepiped?

Ps: Three decimeters.

T: How many cubic decimeters did we fit along the length?

Ps: 3 cubic decimeters.

T: How many rows did we fit?

Ps: One row.

T: Let us put down another row of 3 cubic decimeters. Now how many rows fit on the bottom of the parallelepiped?

Ps: Two rows of 3 cubic decimeters each (Figure 41).

T: What is the width of the parallelepiped?

Ps: Two decimeters.

T: Why could we fit two rows?

Ps: Because the width of the parallelepiped equals two decimeters.

T: How many cubic decimeters are there in each row?

Ps: Three cubic decimeters each.

T: Why could we fit three cubic decimeters into each row?

Ps: Because the length of the parallelepiped equals three decimeters.

T: Has the whole volume been filled with cubic decimeters?

Ps: No, not all of it.

T: How many layers did we put?

Ps: One layer.

T: How many such layers remain to be fitted?

Ps: One layer more.

T: Let us put one layer more. How many cubic decimeters are there in each layer?

Ps: It must be 3 cubic decimeters x 2 = 6 cubic decimeters.

T: How many layers of 6 cubic decimeters each fitted into the parallelepiped?

Ps: Two layers.

T: Why two layers?

Ps: Because the height of the parallelepiped is 2 decimeters.

T: How many cubic decimeters fitted into this parallelepiped?

Ps: 3 cubic decimeters x 2 x 2 = 24 cubic decimeters.

T: What is the volume of the parallelepiped?

Ps: The volume of the parallelepiped equals 24 cubic decimeters.

T: What form does the classroom have?

Ps: The classroom has the form of a parallelepiped.

T: Can we determine the volume of the classroom?

Ps: Yes.

T: With what measures?

Ps: With cubic meters. One would have to fill the class with cubic meters.

T: Is it convenient to determine the volume of the classroom this way?

Ps: No; the desks, table, and cupboard would hinder us.

T: Is it possible to measure the volume of a parallelepiped by another method? When we measured the parallelepiped, we put cubic decimeters along the length, width, and height. Then we multiplied these numbers. How do we determine the area of a rectangle?

Ps: We measure the length and the width and multiply the numbers obtained.

T: How can we measure the volume of the parallelepiped?

Ps: One should measure the length, the width, and the height and multiply the numbers obtained.

T: Danilov, measure the height of the classroom.

Danilov measures the height of the classroom, which is 3 meters.

P: The height of the classroom is 3 meters.

T: And now let us compute the volume of the classroom. The volume of the classroom is equal to 5 cubic meters x 4 x 3 = 60 cubic meters. And now write down in your notebooks the rule for determining the volume of a parallelepiped.

To determine the volume of a parallelepiped one should measure the length, width, and height and multiply the numbers obtained. The rule was formulated by the pupils themselves.

T: Let us solve the problem: "A room is 5 meters long, 4 meters wide and 3.5 meters high. Compute the room's air capacity."

Plan and Solution:

T: What is the air capacity of the room?

Ps: 5 cubic meters x 4 x 3.5 = 70 cubic meters.

T: What have we found out today?

Ps: Today we have learned how to compute the volume of a parallelepiped.

T: By what methods can one compute the volume of a parallelepiped?

Ps: The volume of a parallelepiped can be computed by filling it completely with cubic units of measure or by determining the length, width, and height and multiplying the numbers obtained.

Homework:

T: Every pupil should measure the volume of his room. How will you measure the volume of a room?

Ps: We will determine the length, width, and height and multiply the numbers obtained.

When studying cubic measures in the auxiliary school it is especially necessary to give serious attention to the form of notation. In solving practical problems in the auxiliary school one may encounter very diverse forms of notation just as we found in our investigation.

We recommend that in the auxiliary school only one form of notation be used:

5 cubic decimeters x 4 x 3 = 60 cubic decimeters.

The computation may be done separately in a form like:

5 x 4 = 20,
20 x 3 = 60.

The form of notation "5 decimeters x 4 decimeters x 3 decimeters = 60 cubic decimeters" in general should not be introduced in the auxiliary school since, by its structure, it is not understandable to the mentally retarded child and produces a series of imprecisions and mistakes in solving a problem. The pupils may write it down as "5 decimeters x 4 decimeters x 3 decimeters = 60 decimeters," or may introduce other incorrect forms of notation.

In our practice we encountered cases in which the pupils determined the volume of a parallelepiped in partial calculations. First they determined the area of the base and then they multiplied the result obtained by the height of the parallelepiped. The notation was in a form like:

5 decimeters x 4 decimeters = 20 square decimeters;
20 square decimeters x 3 decimeters = 60 cubic decimeters.

These pupils had come to the auxiliary school after the fourth grade of the public school. In the public school they had been instructed in such a form of notation, and these pupils continued to use this method in the auxiliary school. For the pupils of the auxiliary school such a form of notation is also inadmissible, since determining the volume in a given case is carried out mechanically, and the pupils do not understand the logical essence of this notation. As a result of this notation, the pupils made more mistakes. This notation is introduced only in the upper grades of the public school.

After the pupils have been acquainted with computing the volumes of the parallelepiped and of the cube by these two methods, one should conduct as many practical studies as possible to consolidate skills and abilities in determining the volumes of geometric bodies. The practical projects may be very diverse: determining the volume of the classroom, of a room, measuring the volume of wood, the volume of a pit, and the like.

In measuring the air capacity of a classroom, of a room, and of similar things, the pupils should be told what air capacity is and visually shown with what units one should measure. For this purpose it is good to use a cubic meter with the front face open.

With these pupils it is especially necessary to solve problems in determining the volumes of cubes and parallelepipeds from a model. In this case the pupils solve the problem creatively. For the pupils of the auxiliary school, determining the volume of a cube is more difficult than determining the volume of a parallelepiped. In determining the volume of a cube, the pupils often determine the full surface area, or they even multiply the length of an edge by the number of faces.

In determining volumes the pupils should understand that to determine the volume means to find out how many cubic units of measurement the given volume contains. When the pupils master this proposition, they will not write a final result in the form "48 centimeters" or "48 square centimeters" but rather "48 cubic centimeters."

Determining the volumes of geometric bodies should be conducted in the following basic stages:

a) acquaintance with the concept of the volume of a geometric body--the pupils convince themselves that a volume is the portion of space occupied by a given body or the capacity of a given body;

b) acquaintance with the units of measuring volumes. One should show the pupils the difference between linear, square, and cubic measures;

c) measurement of the volume of a rectangular parallelepiped and a cube by filling a model completely with unit cubes and by directly measuring the length, width, and height and multiplying the numbers obtained;

d) in the process of measuring volumes, designation of a precise form of notation:

5 cubic decimeters x 4 x 3 = 60 cubic decimeters;

e) attention to determining the volume of a cube;

f) consolidation of the conclusion obtained (To determine the volume of a parallelepiped one should measure the length, width, and height and multiply the numbers obtained) in the pupils' memory by their solving practical problems in computing volumes.

On the basis of our experimental investigation and of the comprehensive study of the pedagogical process, we gave, in a definite sequence, certain methodological instructions for the study of geometric material in the auxiliary school.

Conclusion

As a result of our investigation we come to the following conclusions:

1) The Russian elementary school has a rich literature on the problem of teaching visual geometry. The conception of the idea of teaching visual geometry ("children's geometry") for us in Russia goes back to the end of the 18th and the beginning of the 19th centuries. This is independent of foreign authors. An attempt to introduce visual geometry in the school course took place in the 6th decade of the 19th century.

2) The educational programs of visual geometry were rather numerous and proceeded from various methods of teaching:

a) At the basis of the program by V. A. Evtushenko, M. O. Kosinskii, A. Gel'man, and others lay the examination of geometric bodies. In the examination of geometric bodies the concepts of geometric body, surface, and figure, and similar concepts were brought out by the pupils themselves. A group of these methodologists emphasized certain didactic principles as visualization, independent activity, and active participation.

b) At the basis of P. van der Vliet's program was the principle of ground measurement. The basic geometric concepts were elaborated in visual form in the solution of practical problems of measuring the ground.

c) E. Volkov and M. Boryshkevich based their programs on the principle of the pupils' individually constructing the figure being studied. After such concrete study they passed on to geometric abstractions. The authors of these programs considered that the mere contemplation of geometric forms was insufficient. In the latter two systems the process of studying visual geometry was constructed through the pupils' active participation. The independent activity motivated the child. The pupils traced out geometric figures and tried to formulate appropriate conclusions. The study of geometric forms was begun with the line as the simplest geometric form and concept.

3) The basic aspect uniting all these programs was the fact that in the study of geometric material all these authors proceeded from models, independent activity, interest, and the pupils' active participation.

Thus the basic didactic principles of the best pedagogs—K. D. Ushinskii, N. I. Pirogov, L. N. Tolstoi—have been reflected in instruction in geometry.

4) In the prerevolutionary school, lessons in visual geometry were of an episodic character. The teaching of visual geometry was worked out according to the initiative of individual methodologists and practicing teachers. The necessity of introducing visual geometry into the school course was not determined.

5) Only the Soviet school positively determined the necessity of introducing visual geometry into the school course. In the Soviet school of the twenties there were two trends in the teaching of visual geometry; some methodologists proposed to begin the study of geometric material with geometric bodies, others, with the line.

6) Visual geometry is the basis of the study of geometric material in the auxiliary school. The prerevolutionary auxiliary school did not include the study of visual geometry in its curriculum and took as its basis the curriculum requirements of the public elementary school. Beginning in 1927, the auxiliary school began to adopt its own curriculum, and the study of visual material was conducted. The basis of this curriculum was the curricula of the elementary school, with consideration of the work experience of the Moscow and Leningrad Auxiliary Schools.

After the historical Resolution of the Central Committee "Pedagogical Distortions in the Commissariat of Education," when the auxiliary school began to assume its true individuality, the curriculum requirements began to suit the abilities of the auxiliary school pupils.

7) The study of visual geometry in the auxiliary school should be conducted only in the upper grades (5-7); in the lower grades (1-4) preliminary studies should be conducted. The goal of these studies is to sharpen and widen the store of elementary geometric ideas and concepts and to prepare the pupils for the study of visual geometry in the upper grades.

8) The pupils of the lower grades of the auxiliary school should be presented with elementary ideas and concepts about geometric forms: the sphere, cube, parallelepiped, circle, triangle, square, and rectangle.

9) With the pupils of the upper grades one should conduct systematic lessons to widen and deepen the geometric concepts about these same geometric forms; the pupils should also be acquainted with the parallelepiped, cylinder, cone, and pyramid. Moreover, one should present a definite system of knowledge about square and cubic measures. The latter section is especially necessary for the mentally retarded schoolchildren, since it has wide use in practical activity.

10) The study of geometric material in the first and second grades should be conducted according to a plan. The students should learn to:

a) select geometric figures and bodies according to a model;
b) select geometric figures and bodies according to name;
c) name geometric figures and bodies.

It is advisable to begin the selection of geometric forms with the geometric bodies which are most easily learned by the mentally retarded children; it is advisable to begin the initial exercises with the ball as the simplest geometric body and the one most familiar to the children. Then one should conduct the selection of geometric figures. It is advisable to begin choosing the circle as the simplest geometric figure and the one most familiar to the children.

11) In the third and fourth grades the elementary geometric concepts are broadened; studies should be conducted along two lines:

a) comparing geometric figures and bodies (ball and circle, cube and square, parallelepiped and rectangle);

b) comparing similar geometric figures (rectangle and square) and geometric bodies (parallelepiped and cube).

In the process of these studies the pupils look for similarities in diversity and diversity in similarity. Such studies enrich the process of becoming acquainted with geometric forms and promote the development of the child's abstract logical thought.

12) In the upper grades (5-7) the study of geometric material should begin with the simplest geometric concepts, and the teaching should be conducted so that what follows is based on previously known material.

In studying square and cubic measures special attention should be given to the form of notation, to the order of presenting the educational material and to the units of measuring length, area, and volume.

It is necessary first to acquaint the pupils with the concepts of length, area, and volume and then to acquaint them with the units of measuring length, area, and volume.

The studies on measuring areas should be conducted thus:

a) by covering completely with square units of measurement;

b) by dividing up the area of a rectangle or square using one unit of measurement;

c) by dividing into square units with a ruler and counting the number of squares directly;

d) by determining the length and the width (base and height) of a rectangle or a square and multiplying the numbers obtained.

The studies on measuring volumes should be conducted thus:

a) by filling completely with the units of measurement and counting them directly;

b) by determining the length, width, and height and multiplying the numbers obtained.

With mentally deficient children it is especially necessary to conduct more studies on determining the areas of geometric figures and the volume of geometric bodies using models. Such studies promote the development of the creative abilities of the auxiliary school pupil.

13) In the study of geometric material in the auxiliary school it is necessary to use visualization, active participation, and an individual approach to the pupil. In the lower grades it is necessary in the study of geometric forms to use extensive visual and auditory stimulation, and to work with modelling geometric bodies and tracing geometric figures. In the upper grades both in the study of geometric forms and in the study of square and cubic measure, it is necessary to use extensive visualization, modelling, tracing, and surveying. Mastery of geometric material by the auxiliary school pupils will take place only if the solution of vital practical problems is conducted during instruction.

14) Our investigation has shown that the auxiliary school pupils display individual differences in their development; hence, an individualized approach in visual geometric instruction in the auxiliary school is of great importance.

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Visual and Verbal Means in Preparatory Exercises in Teaching Arithmetic Problem Solving*

N. F. Kuz'mina-Syromyatnikova

Introduction

Describing the cerebral hemispheres as "a special instrument of associations and connections, also having higher reactivity," Pavlov indicates that "any disorder in one place must make itself known, affecting the entire instrument, or at least many individual points or parts." The imperfect work of the "special instrument of associations and connections" of a mentally retarded schoolchild also appears in the process of his cognitive academic activity. The teacher will continually notice that in mentally retarded children the formation of new associations is difficult, the shallowest analysis is imperfect and at times, without special teaching methods, is impossible. The thinking of a teacher-defectologist** in such a case is naturally directed toward the peculiarities and difficulties he encounters in teaching the mentally retarded child, and those which he must overcome by special measures.

First of all, it should be pointed out that in all areas of the mentally retarded schoolchild's cognitive activity, more or less pronounced dissociations are observed in the activity of the first and second signal systems. It thus becomes clear how essential it is in schoolwork to organize the coordinated activity of both signal systems in order to provide a more successful assimilation of knowledge. In this outline we will try to examine some of the problems of teaching mentally retarded children how to solve arithmetic problems.

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**Specialist in teaching the mentally retarded (Ed.).

We begin by pointing out that productive activity by mentally retarded schoolchildren in solving arithmetic problems is realized as a result of prolonged, systematic, and, we propose, specialized instruction. Like Pavlov, we understand school instruction in this connection to be the formation of a "long series of conditioned reflexes."

Teaching mentally retarded children how to solve arithmetic problems, despite all the attention teachers have given this task, often does not produce the necessary effect. Many mentally retarded schoolchildren do arithmetic problems poorly, and in the process of instruction do not acquire the knowledge, skills, or habits necessary for practical work.

A lack of research into the specific features of problems confronted by mentally retarded children in mastering arithmetic material, research which could serve as a guide for teachers, must be considered one reason for this. Such work began to appear only in the last decade.[1] Compilation of data characterizing features of problems which mentally retarded auxiliary school students experience with arithmetic material had only been recently begun, and there were still no general theoretical positions upon which a scientific analysis of previously gathered material could be made. This situation was changed after Stalin's work, "Marxism and Linguistics Problems" appeared, and a session was held dedicated to the physiological teachings of Pavlov. Since then, examination of problems in mastering schoolwork--and, as a result, of problems in teaching--has changed in principle. The method of examining data related to pathology has become apparent.

A comparative appraisal of facts obtained while investigating normal and anomalous pre-schoolers is still timely; however, today a solution of problems concerning a system of pedagogical, and, further, methodological modes of teaching children with anomalies of development is not intended to adapt the methods of teaching the normal child to the instruction of the mentally retarded, but rather to develop a system of methods suited to the mentally retarded. This is necessary because the analytic-synthetic activity of a mentally retarded schoolchild, which results in the solution of any arithmetic assignment, is highly distinctive.

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[1] Dissertations of K. A. Mikhalskii, P. G. Tishin, T. V. Khanutina; articles and investigations into the question of mastery of arithmetic material by auxiliary school pupils by N. F. Kuzmina, M. E. Kuzmitska, K. A. Mikhalskii; from psychological works, an article by E. M. Solovev.

In teaching arithmetic, problems may be presented in one of two forms, visual-verbal or verbal alone. Generalizations that the schoolchild meets in solving an arithmetic problem, and which are necessary for solving this problem, appear before him in definite words and word combinations. Numbers, the conventional representatives for concrete quantities, enter into certain correlations only under the direct influence of the verbal part of the problem. The solution of each arithmetic problem, even if it reflects a real counting situation (measuring or weighing), becomes possible only as a result of knowing how to verbalize certain general laws.

Methods intended to develop speaking and thinking in the mentally retarded schoolchild always lead the teacher to the question of the relationship between visual and verbal means of instruction. Beyond doubt, "basic laws, established in the work of the first signal system, must also govern the second, because this is the work of the same nerve tissue." Isolating the second signal system from the first can "distort our attitudes toward reality [3:722]." Thus, only coordinated activity of both signal systems ensures a correct understanding of objects and phenomena of surrounding reality such as relationships among numbers in an arithmetic problem.

We are assuming that to provide coordinated work of the first and second signal systems of a mentally retarded child, particularly in teaching him to solve arithmetic problems, it is necessary to find special modes of utilizing visual and verbal methods of instruction. In this work we are examining only a certain part of the problem of visual and verbal methods; and we are doing this using only a system of prepared exercises in teaching how to solve arithmetic problems.

The Influence of Previous Experience on Solving New Arithmetic Problems

In doing any assignment, the pupil is always to some extent guided by his experience. And the richer and more varied, and, in particular, the more organized this experience, the easier it is to do the assignment. Ivanov-Smolenskii's statement that a child can solve an initially unsolvable problem "by appropriate experimental organization of his past experience [2:19]" is completely valid for the teacher who uses the system of school assignments in teaching. In constructing each new assignment the teacher must inevitably be guided by the exact level of his students' knowledge and by their ability to draw on this knowledge in solving new assignments and exercises. Furthermore, a teacher in an auxiliary school should know and consider the characteristics of his students when referring to and applying his experience in overcoming new difficulties.

The student's previous experience is always to some degree reflected during the course of completing a new assignment, as well as in the results obtained. In some cases this influence is positive; in others, reference to past experience alters the assignment and leads to a wrong solution, and is therefore negative.

When the pupil's experience is too limited and fixed by monotonous exercises, solving new problems is difficult. This is especially striking in the work of mentally retarded students and even of juveniles who have finished auxiliary school. The system of temporary ties that was formed in the cortex during instruction in school is retained and in new activities often appears only in the form fixed by the exercises repeated in school. New efforts and new exercises are needed to lead the mentally retarded student to the solution of new problems.

Our pedagogical investigations convince us that children can solve a new problem most often when, besides oral instruction, they are given an example of a concrete activity from their past in the presence of some visual teaching device. When verbal and visual means are not properly combined, instruction does not provide the students with the necessary knowledge and, still more important, does not provide the necessary and productive independent activity needed by the child or adolescent in solving various new problems. The younger the child the more visible his helplessness in a new situation.

Let us present some examples.

Felix L., 11 years old, first grade. Studying under several successive tutors, the boy learned to count. Frequent repetition of the numbers from one to ten and back again brought him to the point where the problem of counting objects, drawings, points of a number figure, and other things he had not already counted during instruction, left him at a dead end. Not only was he unable to determine the number of objects in a given group, but he saw no need to determine this. Counting meant naming words, it meant reciting all the words—the numbers from one to ten and ten to one.

Yana M., 10-1/2 years old, first grade. Having learned as a result of frequent repetition that a cat has two front paws and two hind paws, she could still not answer when asked how many paws a cat has in all. Furthermore, she asserted that the question itself is impossible.

"What are you asking? What were you thinking? Don't you know, a cat doesn't have 'all' paws. It has two front and two hind ones."

And even after directly counting a cat's paws Yana still didn't agree, and stretching a rubber cat, maintained, "Why do you need to count like that? Here they are--two front and two back."

Yura L., 11 years old, second grade. In solving a two-operation arithmetic problem (augmentation by several units and finding the sum), Yura, whenever he had trouble, resorted to visual representations of the problem which had been used in his previous schoolwork. Solving an orally presented problem, he changed his solution several times, getting it wrong every time until the investigator let him count the groups of objects. Twice Yura read the problem. Then he recited.

Text of problem	Yura's interpretation of problem
A hen had 12 yellow chicks, and 4 more black chicks than yellow ones. How many black chicks did the hen have?	A hen had 12 yellow chicks, and 4 more black ones than that. How many black ones?

Yura's solution	His explanation
I. $4 \div 2 = 2$. II. $12 \div 2 = 6$.	So there are yellow ones and black ones, and 4 of them. There were 12, they were black and yellow. You have to divide into equal parts.

When asked, "Why divide when they say in the problem there are 4 more black ones than yellow ones?" Yura replied, "But we're learning how to divide." Classwork experience was applied to solving a new problem which had nothing to do with division into equal parts. By sketching the problem, Yura saw how to solve it and carried out the solution correctly. He met willingly the suggestion that he sketch the problem, and his interest in it was heightened.

"What should I draw first?" asked Yura. The investigator suggested that he examine the problem by himself. Work proceeded in an exceedingly interesting manner. Yura drew and wrote, continually consulting the text of the problem. "We have 12 chicks." (He writes the number 12 and draws 12 sticks.)

"Here they are all in a line. What kind are they? 12 yellow ones, but I don't have a crayon. But so what? I know and you'll understand. Look, these are the 12 yellow ones. And there are 4 more black ones. How many?" (He reads, "4 more black ones than yellow ones.")

"There must be 4 more. Gee, how long the line will be with 12 and then 4." (He draws only 4 sticks.)

"Here are the 4. The 4 more. The first 12 are yellow, and these are black: 12, 13, 14, 15, 16." (He reads and writes 12 + 4 = 16 on the right and the number 16 on the left [see Figure 1].) He was very happy that the problem was solved.

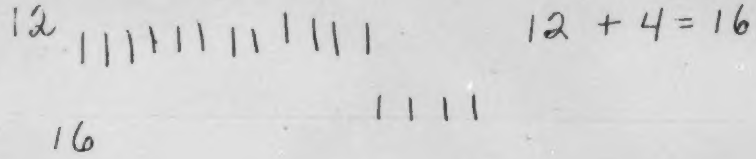


Figure 1

The assignment was complicated. The teacher said, "And what if the problem were, 'There are 12 yellow chicks and 16 black ones. How many chicks does the hen have in all?'"

The boy looked at his drawing and talked and pointed. "That's easy now. Only you have to say this first (draws a Roman numeral I, by his sketch) and this will be the second problem." He writes the Roman numeral II, then 12 + 16 next to it, and begins to solve. He writes the answer 18 (Figure 2).

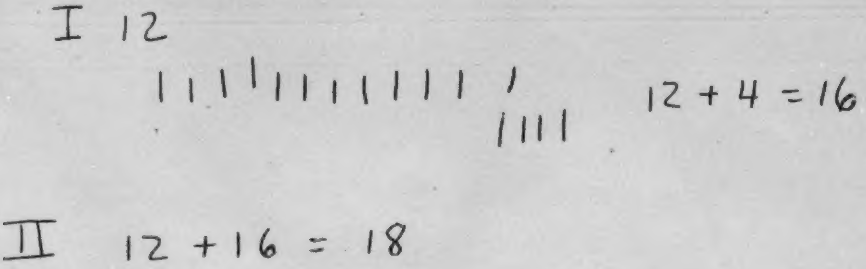


Figure 2

Yura's answer was not accepted. "It's right, no, it's right. 12 and 6 are 18," asserted Yura. And when it was pointed out that he had to add 12 and 16, not 12 and 6, he answered calmly, "But we don't know how. We know how to do 6."

He was then given help in solving the problem.

Teacher: Can you add 10 and 10?

Yura: Yes. 20.

T: And 2 and 6?

Y: Yes. 8.

T: And 20 and 8?

Y: We can't. We haven't learned yet.

T: And what if I help you? I will give you 20 kopeks and then 8 more.

Y: We know that. 28 kopeks.

T: It's the same here.

Y: No, this isn't kopeks, this is chicks.

T: And have they solved the problem?

Y: They don't know; we know.

Yura did not limit himself to one solution. On a scrap of paper he drew two circles, and in each he wrote "10 kopeks" and calculated: "10 kopeks and 10 kopeks are 20 kopeks, and 2 and 6 are 8." Writing, he says, "20 kopeks plus 8 more will be 28 kopeks in all. That was easy, and we'll write 28. We must cross out 18." (Figure 3).

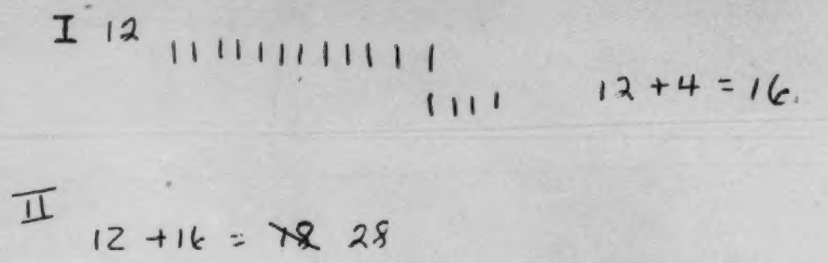


Figure 3

Yura tore up the scrap of paper with the auxiliary drawing, saying, "This was for us." As to my facetious question whether we would tell the chicks that we had counted kopeks and found out how many chicks there were, Yura understood perfectly and answered, "Kopeks are easy, but there are no chicks here; they're in the numbers."

Yura solved the following problems independently, making drawings for each one. In class at this time they were solving problems by dividing by 2. We gave Yura division problems, not division by 2 but by 3. Until he had made a drawing of it, Yura solved the first problem by dividing by 2 (instead of 3). Only after illustrating it in a drawing did he get the correct solution. The units in this problem were needles that had to be distributed into three packs.

In solving the next problem Yura, not finding the answer immediately, started sketching. The units in this problem were nuts which were to be distributed into 3 boxes. Yura began to draw needles. He stopped. "I am drawing needles and I should be drawing nuts." He drew 12 little circles (nuts) and 3 quadrilaterals (boxes). Then he started solving the problem. "We must have boxes. If each has 3, there will be 9, and there have to be 2." He drew 3 circles in each quadrilateral and crossed out 9 circles. "There are still 3 nuts. There is one for each," and added a "nut" to each "box." He wrote down 12 ÷ 3 = 4 (Figure 4).

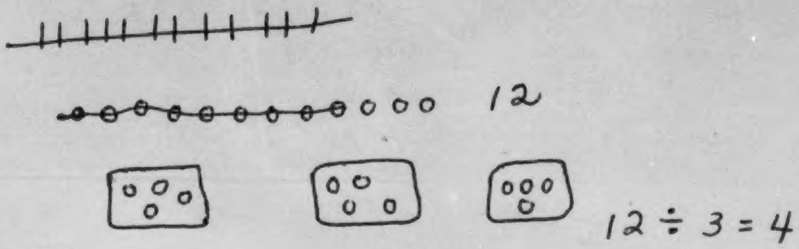


Figure 4

For one of the following assignments Yura was to solve a problem difficult to represent. The problem was taken from a second grade text. Even after reading the problem twice, Yura could neither solve nor repeat it.

Problem	Yura's interpretation
Five identical loaves of bread were baked from 10 kilograms of flour. How many kilograms of flour were used for each loaf?	Five loaves of identical kilograms were baked from flour. How much flour was used for the loaves?

His solution

10 - 5 = 5. And 5 loaves were left.

Yura did not obtain the necessary solution even after analyzing the problem, since he had no understanding of the kilogram as a unit of weight.

Yura asked for permission to draw the problem. "Then I'll see it and solve it." In drawing and solving the problem, Yura resorted to his previous practical experience: "With 3 kilograms in each, we bought 2 packages, and here's 10." He drew 3 quadrilaterals (packages) and wrote the figure 3 in each one. "In 3 there are 9 kilograms, and we need 10. There's still a little one left." He drew and wrote a small number 1. He looked at the drawing for a long time and came to the conclusion: "There can't be 5 loaves, we have 4 kilograms."

He looked at the problem. "We have to draw 10," and he draws 10 squares. Again he referred to the text of the problem. "And we must have 5 more loaves." He drew 5 "bricks." "We must put flour on the brick." He drew a little quadrilateral in each "brick." He crossed out 5 of the 10 squares (kilograms). He then drew another quadrilateral in each "brick" and scratched out the remaining 5 squares (kilograms). "There will be 2 in each brick." (Figure 5)

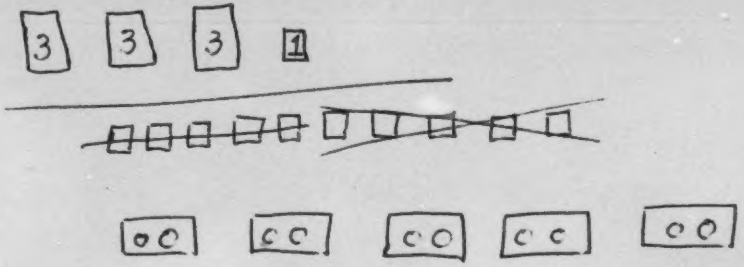


Figure 5

He was asked to explain the terms of the problem using the drawing. Now his interpretation of the problem was much more complete: "There were 10 packages with kilograms of flour. They baked 5 bricks of bread. How much flour is needed for each loaf?" The solution 10 ÷ 5 = 2 was correct, although this solution was expressed in the form of an arithmetic sentence without corresponding units. When requested to explain the problem in writing, Yura obtained the correct answer "2 kilograms for one loaf."

Elena M., having finished auxiliary school and having started night school in the fifth grade, came to us complaining about the difficulty of her arithmetic lessons. It was particularly hard for her to solve arithmetic problems. Numerous difficulties were explained in class, but her basic problem was an inability to apply what she had learned in school to new and unfamiliar conditions. The slightest variation from the usual way of doing a problem left her at a dead end. She partially understood the assignment to count by groups. Eleana reproduced the numbers from 1 to 20 successively.

An assignment to count by fives with the example "5, 10, ...", Elena did this way: she named in succession all the numbers in the series, 1-20, raising her voice on 5, 10, 15, and 20. She counted by 2's the same way. She did it correctly only after a concrete example of counting by groups was given. She was asked to recall how change is counted out by pairs of kopeks.

Teacher: "How would you count, if I put down two kopek coins in front of you?" and further: "How would you count if I put down five kopek coins before you?"

It is interesting that solutions were realized only as an imitation of a concrete activity, that is, counting coins.

She carried out an assignment in counting by numbers larger than five by reproducing the lines of a multiplication table. Elena did not grasp the model of counting by sixes--6, 12, 18... "No. We must do it like this: 6 taken twice is 12, 6 taken 3 times is 18, ... " she said, reporting every line of the table. When asked to write the answers she had obtained, she wrote arithmetic lines from a multiplication table.

When asked to count by sixes by marking them off one after the other and writing down the answers, she used addition signs and equal signs, that is, she translated the assignment into an arithmetic example. The new, conventional form of writing was not acceptable to the examinee even when it was suggested that she solve the assignment with multiplication. Every line of writing underwent a change; the figures were joined by operation signs and without fail an equals sign was put into every line.

Generalizing the examples given here we can conclude that students of the lower grades of the auxiliary school often try to apply their previous experience to new problems without any modification. They find ways of solving a new problem by relying on visual aids which they used before in solving similar or identical problems. Older pupils also had to rely on past experience. But concrete examples (kopeks in counting chicks with Yura, coins in counting by groups with Elena) are not always needed for this group; more often this is a pattern of the usual way of writing. For example, Elena inserted addition signs, multiplication signs, and equals signs to carry out the assignment of counting by groups.

We find an explanation of this fact in one of Ivanov-Smolenskii's works. He writes:

> With frequent repetition or exceptional intensity of certain conditional situations, structures of relationships which frequently dominate and, because of brake induction, suppress other cortical activity, are created in the large cerebral hemispheres of man [1:403].

The Influence of the Verbal Stereotype in Solving Arithmetic Problems

Knowing how to solve complicated arithmetic problems presupposes a knowledge of how to solve simple problems, which enter into any complex problem as parts in certain interrelations. However, solving even a simple arithmetic problem requires varied abilities of the child; he must discover the relations among numerical facts dependent upon the problem's question. But mentally retarded schoolchildren, unless provided with special exercises, can not figure out the function of the question, and approach it, influenced sometimes by their last experience, and, in other cases, guided by their reliance on individual elements of the problem. In both cases an incorrect application of past experience is the cause of erroneous solutions.

One can notice, even in first graders, a tendency to be guided by external elements which causes them to choose the requisite arithmetic operation. Examples of these external elements are frequently repeated "supporting" words in the problem as "the total will be"; "there were in all" (stipulating a solution by addition); and "will remain" (indicating subtraction).

A child's experience, obtained as a result of solving numerous problems on finding sums or remainders, causes him to choose the correct arithmetic operation. But this experience is inadequate and the problem is solved incorrectly if the customary supporting word is missing from the text of the problem. In this case the child's experience serves as a guide, but has no direct relationship to the essence of the problem—its content. The child solves the problem by manipulating numbers. He is used to seeing that for subtraction the larger number comes first. He knows that he is supposed to take the smaller number away from the larger.

In the terms of the problem, the larger number is first, therefore one can and must solve by subtracting. The reverse sequence impels the child toward a solution by addition. Two identical problems with different number sequences and arrangements were shown to the students.

1. A boy had 5 kopeks. His brother gave him 3 kopeks. How much money did the boy have in all?

2. How much money does a boy have if he had 3 kopeks and his brother gave him 5?

The first problem was solved. The second caused some difficulty. The pupils concentrated on the numbers here. The second problem was not solved because there were no words that definitely elicited a choice of the necessary arithmetic operation--no "than he had" or "he had in all."

Children consider it quite legitimate to answer the problem's question by repeating the terms of the problem: "He had 3 kopeks and 5 kopeks"; "The boy had 3 kopeks and got 5 kopeks from his brother."

Comparing the terms of the two problems, the children did not agree when it was pointed out that the problems were the same. They contended that in the first one, "They ask how much he had in all," therefore, "you must add." Supporting words had caused them to add. The familiar words, "he had in all," served as directions whose result was the correct solution.

The absence of supporting words which organize the usual activity of the mentally retarded schoolchild leads to an incorrect solution of the second problem. The main point of the problem is understood correctly when coins figure in its terms or when a problem is accompanied by visual explanations, that is, when the situation in the problem is made concrete.

In the second grade, problems about differences were solved. In individual assignments the children were given, one after the other, the following two problems. Terms of the problems were read by the children.

1. There are 11 white rabbits and 9 gray rabbits in a rabbit hutch. How many more white rabbits than gray ones are there?

2. A hen has 11 white chicks and 9 gray chicks. How many chicks are there in all?

The students solved both problems the same way (11 - 9 = 2, there are 2 more rabbits, 2 more chicks). Furthermore, previous connections supported by the first problem were found stronger.

These same problems were given to the students to solve a second time. However, the conditions of the problem involved a different color. The children felt a need to compare tables. A question arose: "Why are they alike (the first parts of the problems) here, but here they're colored differently?" (the questions of the problems). The questions were reread. But rereading the questions did not lead to the correct solution either. Certain children, noticing the colors in the construction of the problems, did not discover their essential difference, "since there are rabbits here and chicks there," and solved the problem the same as before. Others noticed the difference and explained correctly: "Here it's 'how many in all?' but here it's 'how many more white rabbits than gray ones?'" In spite of this, they repeated the erroneous solution: "Still everything's the same here [the problem's numbers]."

For certain children, consistency in the number arrangement was decisive in choosing the arithmetic operation: "You take away, since 11 is bigger than 9." A problem similar in the text with the same numbers in reverse sequence (9 gray and 11 white) was solved by addition.

A single word in a problem's conditions or question can cause a mistaken solution. For example, the students solved two problems with identical solutions differently simply because one contained the word "more" and the other "fewer."

1. There are 9 short nails and 6 long nails in a box. How many more short nails than long nails are there?
2. There are 9 short and 6 long nails in a box. How many fewer long nails than short nails are there?

They solved the first one by addition, the second by subtraction.

Expanding the text of the problem in words which focus attention on quantity relations (which kind of rabbits are there more of, and how many more?) improves things. Having established that the quantities were unequal, the children nonetheless did not seek a new, correct solution, and the situation remained as before.

Only after each problem had been made concrete by examples of calculation (cardboard chicks, nails), and after the children looked at and compared the unequal quantities while getting direct oral instruction from the teacher, did they go on to the correct solution.

It should be noted that the children who did not receive instruction did not get the right results, but not because the concrete examples distracted the children. No. The children acted purposefully. Each strove to solve the problem. But in any case, the indicated problem with concrete objects was solved without the teacher's instruction by isolating the extra objects and counting them (how many more). Putting 9 short nails in a row and a row of long nails right under them, the child immediately determined that there were more short nails. It was not necessary to ask "What kind of nails were there more of?" The following questions were absolutely necessary. Here one may proceed in one of two ways. a) Ask the child to answer the teacher's question, or b) refer him to the problem, that is, tell him to read the problem himself and answer it.

The second way is better since the child is not distracted from the text of the problem. Solving the problem this way he still is directed by verbal instructions; but independent activity—reading the question and relating it to the visual way of grouping extras, recounting the extra objects--all this facilitates problem solving. The child gives the right answer: "There are 3 more short nails." The words, "than longer ones," most often are not part of the child's answer.

In this case the child did not understand the essence of difference relationships, and visual aids did not help; furthermore, the given situation prompted the child to isolate the extra three and not subtract six from nine. This possibility could only be discovered by uniting verbal and object activity. The teacher definitely played the leading role here. He directed the child's activity with his assignments and questions.

Teacher: What kind of nails were in the box?

Pupil: Long ones and short ones.

T: How many short nails were in the box? Look at the problem and answer.

P: [The pupil finds the corresponding part of the problem and reads it.]

T: Put the short nails in a row.

P: [He does so.]

T: How many are there? Count them and tell me.

P: [The task is carried out.]

T: How many long nails were in the box? Take the necessary number of long nails and arrange them in another row under the short ones. Explain the problem and point out everything you talk about.

P: [The task is carried out.]

T: What does the problem ask?

P: [The pupil reads the problem's question.]

T: Repeat the question and point out everything that's asked.

P: How many more short nails [points] than long nails [points] are there?

After this he was told to write out the solution. He didn't know how, but he no longer indicated addition as a means of solving.

A picture of the problem makes him thoughtful and questions arise. "How? And how many more? Three more, maybe?" All this tells us that his thought has awakened. Slowly but definitely he began to free himself from the word-stereotype which controlled him and was a hindrance to solving the foregoing problem. Subsequent activity was always directed by the teacher's assignments and questions. It was concluded when the student gave a verbal account, interpreting the problem's conditions while pointing out each group in the sample problem, telling how to solve, and giving the full answer.

The Guiding Role of the Word in Visual Instruction in Solving Arithmetic Problems

The employment of visual aids in teaching auxiliary school pupils is imperative only when their use makes it possible to reveal the essence of some assignment, the essence of the question being resolved. When organizing school activity around visual aids, the teacher should ensure attentive guidance until the pupil is able to consciously utilize them independently. In teaching schoolchildren to use visual means, the word, the teacher's speech, relying on personal demonstration, describing beforehand, and organizing the pupil's demonstration plays the leading roles.

Incomplete forms of the activity of the mentally retarded schoolchild in general and his academic activity in particular are often observed by the teacher. In implementing a system of exercises and in organizing the academic activity of the pupils of this school, it is especially necessary for the teacher to protect the pupils from incorrectly completing assignments.

Pavlov, revealing the essence of intelligence, defines it as "thought in action," and as "a series of associations that derive partly from past experience, partly right now, before your eyes, and before your very eyes are combined or added into a positive whole, or, conversely, are gradually impeded, leading to failure [4:430]."

The pupils' mistakes in completing arithmetic exercises stem very often from the fact that the teacher did not check to see which associations--correct or incorrect--and in which combinations--correct or incorrect--are consolidated in the child with the initial demonstrations and explanations of the study material. Organization of the pupils' school activity around visual aids plays not the least important role here. Failure in arithmetic instruction is often the result of the teacher's giving his pupil some aid or another, but neglecting to teach him to use it and neglecting to watch the pupil until he finishes working. He does not replace one aid with another at the right time. He does not seriously consider his own instructions when teaching the child to work with an aid, i.e., he explains with words alone, or only demonstrates how to work. A one-time demonstration--and a personal demonstration in general--is insufficient. The teacher must watch how the child himself begins to work independently with the aid and use the instructions as a guide. The teacher must hear out the child's story of how he works with the visual aid.

To illustrate: Vitya E., a second grader of the auxiliary school, was described to us as incapable of being taught arithmetic. The notes in his notebooks would be proof of that. Indeed, an examination of his notebook could lead one to conclude that the boy had not mastered the study material. All solutions were incorrect. Moreover, more profound examination of the mistakes did not reveal any underlying regularity. The results of repeated solutions of one and the same problem were different, almost random. There were no incorrectly consolidated devices for counting. Naturally, we had to see how the boy counted.

In solving each problem given him, Vitya resorted to his fingers, but what he did was not counting on his fingers. It was only sorting out his fingers--not counting. Not one quantity coincided with a number. It was ascertained that the boy could not use other devices for counting either.

In individual lessons with Vitya we decided not to give him anything which would cause incorrect solutions. We began the work using dominoes. Vitya knew these numerical configurations because, as we learned in conversation, he often sat with his friends and watched them play. He correctly named all numerical figures. However, he flatly refused to make up a problem on addition using a domino with two quantities of dots. But when told that he was not released from the assignment and had to do it, Vitya, ignoring the instructions, began to count up all points on the domino. A repeated instruction, supported by the solution and notation of a problem on numerical groups of dominoes, was listened to and the process of the notation procedure was followed. After this Vitya asked that it be repeated. He listened and followed the demonstration accompanying the explanation very attentively. He said that he understood the assignment. He wrote down a problem and did not count the dots in each figure. The solution occurred after he counted all the dots, from first to last. When asked what had to be done, he answered, using a demonstration, "A problem must be made. How much all together, three here and two here." And only after the questions and demonstration accompanying them did he give the conditions of the problem: "We have to find out how many dots in all, in the first one there are three dots, and in the second, two dots."

After joyfully composing a solution of three more problems, he set about working independently on composing and solving arithmetic problems. Eight problems were composed and solved with no mistakes. For the boy, success was unexpected. He voiced the desire "to solve another whole column." And another eight problems were solved without error. "So that's how! And with my fingers it was all mistakes. I've learned with dots."

The assignment was made more complicated. He was asked to compose an addition problem, given groups of three numbers. At first two dominoes were given him, then three. Until the assignment had been consolidated with a demonstration and arithmetical notation, the boy's activity did not commence. Oral instructions without an accompanying visual form of the assignment were rejected. "I don't know, I'm afraid there will be a mistake." However, after a demonstration and an explanation of the assignment, the work was completed successfully. As the work progressed, one domino was replaced by another again and again, and Vitya quietly solved each new problem (the first and second assignments were alternated). The child's activity was wholly conscious. However, his ability to recognize the assignment (with two or three quantities) was impaired as soon as a blank domino was put in place of a domino with quantities. The boy did not throw it out, he did not refuse to compose and solve a problem, but from the empty field he gave the example 2 + 1 = 3. When asked "How's this? Where's 2 and where's 1?" he calmly answered "There was a domino like that, maybe." When given a domino with one numerical group of dots, Vitya gave no example. He firmly exclaimed: "There can be no problem because there's only one number; you have to substitute something."

We went on to play a game. Before composing a problem, he had to find whether a sum could be obtained. Dominoes with one, two, and three quantities and blank ones were put before the child. There was no definite order in which they were shown to him. The boy worked with interest. At first the work proceeded very animatedly.

"No, no," Vitya shouted happily. "You can't—there's no number. Two numbers. Here is a problem. There is no problem—only one number is here."

The composition and solution of problems proceeded totally consciously. Only consciousness could lead to those correct answers which the boy gave to questions of the possibility and impossibility of composing an arithmetic problem. ("Without numbers you can't; with one number you can't; you have to have two numbers, and maybe more.")

The child's newly obtained experience allowed him to make these generalizations. But the boy could not transfer these generalizations to the solution of a slightly expanded problem. To compose a problem, "you have to have two numbers, and maybe even more," says the boy. When the experimenter said "And even more?" the child responded correctly, "You can have three numbers." He composes these problems with two and three items. The question of the possibility of composing a problem with four items remains unsolved: "I don't know, they didn't show me." And when we placed before him a domino with two quantities, or four dominoes with one quantity each, or three dominoes, one having two quantities and the others having each one quantity, Vitya came to a solution of the possibility of such a problem: "You can do it, but for me it's hard to count." He gives the correct notation and refuses to solve it, fearing to err. Again the solution of the problem had to be facilitated orally and visually.

The boy firmly answered all questions concerning the possibility and impossibility of forming an arithmetic problem, but in the actual solution he was still far from independent. The boy's helplessness was clearly evident whenever he had to pick out the dominoes with whose help composition and solution of a problem would be possible. The boy was always requesting that he be given only the requisite dominoes: "It's hard for me, the teacher must give them to me."

The assignment to compose and solve a problem from four dominoes--one was blank, one had one quantity of dots, and two were full--was completed only after hesitation. The child sorted the dominoes into a row. He looked at them for a long time and moved them around.

"Oh, I've failed! Go away. There's nothing on it. You can't do it without numbers." He takes the blank domino from the row. He looks for a long time, takes the domino with one quantity and puts it beside the blank one.

"You can't do it with one quantity." But he still was not satisfied with this. "Maybe there'll be another one. Did we have them with one? Impossible with one. But we had them." He puts the domino aside.

He begins the notation and solution of the problems. He gives two problems with two items (each with quantities from one domino). He writes down the solution of a problem with four items. He counts them. He counts the items in the first examples. "What did I leave out? I don't know what I left out." He wavers. He takes the blank domino he had set aside. "Impossible with one. But did we have such?"

A positive answer makes him think. "Impossible with one. Where should I put it?" He puts it by the blank one. He takes it away. He puts it beside two full dominoes. "Many like this." He counts the groups. "Oh! So many! We didn't learn about these!"

He reselects the dominoes. He moves the right-hand one (the one with one quantity) away. He moves it back again. "It doesn't work. Impossible with one."

Advice is given: "Take away either the left one or the middle one." He moves away the one on the right then moves it back. "Take which one? The left or the middle? Which one?" He moves the left one away. For a long time he looks at the two remaining ones and is silent. I ask him to take a pencil and write out the solution. "But is there one?" He writes sluggishly.

After writing down the two numbers, he inserts a plus sign and very joyfully shouts: "It's possible." He writes and adds the items. "You can do it with three groups." He solves the problem.

When the work was concluded he wanted to be assured that all assignments were done correctly; and when he was given approval, he went away happy.

At one of the following lessons, Vitya was asked the already familiar questions: "Can one compose a problem without numbers, with one number, with two or three numbers?" He answered all the questions correctly. He undertakes the solution of new problems. Vitya asks permission to use the dominoes "so I can count right." The process of the work follows.

I. A problem with two numbers is given. Vitya is asked:

1) to repeat the conditions of the problem without using the dominoes;

2) to repeat the conditions of the problem by first taking from among several dominoes the domino having two quantities of dots corresponding to the numbers of the problem;

3) to tell the path of solution and solve the problem orally.

Although in the first case Vitya was most often unable to cope with the assigned task, in the second, using the group of numbers as a support, he was more successful at repeating the conditions of the problem. But here too the teacher's guiding word was necessary: "We will count ducks. These dots are the ducks." Or: "We must count kopeks; your dots are kopeks." The boy accepts these conditions and solves the problem. Support from numerical groups (dots) promotes both the repetition of the conditions of the problem and its solution.

II. A problem without numbers is given. Vitya, listening to the problem, turns to the dominoes and, finding a blank, goes back to review the conditions; only after this does he firmly and confidently announce that it is impossible to solve the problem because there are no numbers.

III. A problem with one number is given. Vitya again singles out one of the dominoes with one quantity of dots, refuses to review the problem, and says that it cannot be solved: "It's impossible, there's only one number."

In the examples cited, verbal instruction alone did not promote solution of the problem. Whenever Vitya was given a new assignment, he needed to rely on the experience of a previous activity, on the visual image of a number which was expressed by a numerical group of dots. Even when a numberless problem was given, a statement of its impossibility came only after he had held and seen the blank domino.

One must not overlook the fact that a conclusion was drawn only as a result of the guiding role of the oral instructions and not as a consequence of merely presenting dominoes to the student.

The first help usually given to the pupils in arithmetic lessons are in the form of concrete objects. This is quite correct, since any integral set can be formed only through uniting objects that can be counted.

The concept of number originated in the real world, in surrounding reality. Numerical groups in all cases are some combination of a quantity of actual objects. Hence, objects must be used to teach children to count. Only on this condition can one present the concepts that 1) any quantity is formed as a result of the unification of concrete objects of counting into one group; 2) various objects may be counted into each of the quantities; 3) a quantity, no matter what objects of counting it may be formed from, is constant, i.e., it always contains a specific number of objects.

When children are forced to work for a long while with the same objects of counting, their experience is restricted, and the children's development is retarded. This does not promote the understanding that counting and arithmetic operations are possible only with homogeneous objects. The data of our observations, confirmed by the materials of N. A. Solomin's dissertation study speak very convincingly of this.

Let us cite an example. Pupils with different teachers in two first grade classes solved the assignment, which was to draw a quantity of objects, in different ways, corresponding to each numeral shown to them. Where the counting material was constant (the teacher used colored circles and numerical figures again consisting of circles), the children solved all the proposed problems in the same way. Under each number the pupil drew the corresponding quantity of circles. The explanation that one may draw different objects and that these objects must be the same only within each group was not understood by the children. The solution again followed that same path familiar to them. Whenever the children tried a new approach to solving the problem, the solutions went against the basic requirements. The quantity of objects which the children drew corresponded to the indicated number, but the principle of the homogeneity of objects was distorted. Under the numeral 2 they drew a fir tree and a mushroom, a star and a girl. Under 3, a house, a ladder, and a flag; a window, a ball, a house; etc. Although the explanation of the solution of the assignment was repeated several times, the children solved it in their own way. The children did not understand that, in counting, only homogeneous objects can be included in each of the quantities.

The verbal instructions alone, even if repeated several times, were essentially of no help. The children's previous experience came to the forefront, characterizing definite forms of instruction. The child understood the problem only if it was presented through a demonstration and an explanation of what was required of him.

But even in those cases in which the children understood the problem, its proper solution was possible only under direct influence of instruction. As soon as the teacher was silent, her control over the pupils' independent activity disappeared. They again gave incorrect solutions. Original explanations or explanations not reinforced by exercises were few. Previous experience influenced the child, and his activity remained stereotyped, despite the new instruction. Monotonous verbal repetitions and monotonous activity with the same visual aids influence formation of a stubborn verbal stereotype. The mentally retarded schoolchild, just beginning his education, cannot alter habitual operations independently, despite new instructions.

Selection of visual means of instruction should not be based solely on the suggestions that the teacher finds in the methodological handbooks on the relevant section of the school curriculum. The teacher must always provide for the selection of an aid in accordance with the specific problem presented by the instruction. Each time he must evaluate the level of the pupils' knowledge, and he must be aware of the difficulties experienced by each of them in studying the material. One must also know how the selected visual device can contribute to overcoming these difficulties and to the pupils' best mastery of knowledge. In using manipulative aids early in instruction, one must be very critical of their selection, taking into account the concrete problems to be solved with them.

If we look at initial instruction in counting, we cannot help noting that too much time is devoted to work with separate objects, each of which appears as a unit of counting. Group counting is postponed, a delay which restrains development. In individual cases, counting by units is unnecessarily extended; the pupils, becoming adults, refuse to count on their fingers or with sticks, even though they know no other aids and devices for counting.

The Influence of Knowledge of Arithmetic in Solving Arithmetic Problems

Our pedagogical investigations, organized as individual lessons, were conducted with below average, lagging pupils who often lacked abilities for learning arithmetic.

In all cases we preliminarily established the extent to which the pupil was able 1) to read, to repeat without questions, and, using questions of the conditions of the problem, to tell the method of solving the problem after analyzing it, and, finally, to solve the problem without using visual aids; and 2) to do the same thing employing various visual aids and visual devices of graphic notation.

Before beginning the work, we carefully ascertained the state of each pupil's skills in calculation technique, since the absence of these skills or incorrectness in methods of calculating always leads to incorrect solutions of the arithmetic problem.

In most of the children who could not solve the arithmetic problems, there were also other difficulties in arithmetic instruction. Very often the weak ability to differentiate and the weak analytic-synthetic activity of the mentally retarded child was only slightly considered by the teachers. As a result, many pupils were unable to master the principle of decimal calculation and, therefore, could not give an analysis of the decimal composition of a number; i.e., they could not break it down into series.

The explanations which the teacher usually gave the whole class when studying numeration, the demonstration, and even the work conducted for a while with visual aids left no firm traces in individual children. And this caused their helplessness in completing the calculation exercises.

We did not attempt to reteach such pupils in individual classes. Having ascertained the basic difficulties, we offered the pupils a series of devices for making visual graphic notation, which promotes independent work, on whose basis the solution of the problem was made possible.

We present some examples.

1. Raisa Sh., age 12. Third grader in the auxiliary school. She correctly analyzes a two-digit number, determining the quantity of tens and units in each part. She knows the digit's place in the series. In analyzing the number 29, she says, "29 is two tens and nine units. Two tens because it is written in the second place, and nine units because they are in the first place." She cannot add a two-digit number and a one-digit number. There is no clarity in the order of addition; the units' number is sometimes added with the units and sometimes with the tens of the first item. Information that aids other children in their work just confuses her.

Raisa knows that "It's easier when you first have to add a little number to a big number." Raisa solves the problem 31 + 6 thus: "You have to add one to six because one is smaller, it's easier." She obtains 73. "To six add three tens because three is smaller, and if you add on to three it's very hard." She obtains 91.

Raisa also knows that in addition a larger number is obtained, and in subtraction a smaller number results. "27 + 2 = 29. Why does it have to be 29? You have to add 2 to 27. Indeed, 2 and 2 will be 4. Is this true? And we still have 7, which makes 47." And again: "Add 2 to 27. We have seven units and another two little units, nine in all, and you have to write down another 2. You get nine tens and two units." When asked why she obtained that, Raisa answered: "Because you have to add units to units and tens to tens." When told that the solution was incorrect, Raisa began to argue and presented a proof: "You can check it. If we add, the answer will be bigger, and if we take away, smaller, so 27 and 2 give 92. Which is larger? We got a larger number, and every one we had was smaller [indicates the numbers 27, 2, and 92]. Correct!"

An attempt to give her visual aids evoked a protest: "I'm in the third grade. In the first grade I got fed up with sticks and blocks, and I won't count on an abacus, I did it in second, but I'm in third now, I understand everything." New visual devices had to be sought. Raisa was given written work.

1) Numbers in the problems were written with pencils of different colors: tens in one color, units in another. Before the problem was solved, Raisa analyzed it. "27--there are two tens and seven units in this number. And this 2 is two units, there are no tens here. You have to add units to units [demonstrates]--which is nine and two tens. You get 29." In this way she solved several problems: 27 + 12, 36 + 23, 15 + 5, and others. Raisa makes the notation of the number obtained in the results with the two colors--tens in one color, units in another.

2) Numbers in the problems were written in one color. Before solving, Raisa was asked to indicate the order of addition with colored areas. Raisa asked to be given the directions again and to be shown how she must work. After this she told and showed everything herself. When told she was right, she began to work. She described everything she did. After she had solved several problems, the arrows were replaced by brackets. Raisa looked attentively at the notation and concluded that "This is like with the arrows." She worked assiduously and for a long time. She asked to be given more new problems. When she received one, she asked us to hear it and look at it, to check that everything was correctly marked.

Models of Notation

Assignment	Completion
27 + 2 =	$2\overline{7 + 2} = 29$
36 + 21 =	$\overrightarrow{36 + 21} = 57$
15 + 4 =	$1\overline{5 + 4} = 19$
12 + 45 =	$\overline{1\underline{2 + 4}}\underline{5} = 57$
44 + 32 =	$\overrightarrow{44 + 32} = 76$
35 + 4 =	$3\overline{5 + 4} = 39$

In solving the problems on addition involving regrouping, individual instruction was again required. Using colored notation again, we utilized in all cases "open," detailed notation.

Models of Notation

Assignment **Solution**

$$37 + 45 = \underline{\underset{\bullet}{3}0 + 7} + \underline{\underset{\bullet}{4}0 + 5} = \underset{\bullet}{7}0 + \underset{\bullet}{1}2 = 82$$

$$56 + 17 = \overbrace{\underset{\bullet}{5}0 + \underset{\bullet}{1}0} + \underbrace{6 + 7} = \overbrace{\underset{\bullet}{6}0 + \underset{\bullet}{1}3} = 73$$

$$29 + 26 = \overbrace{\underset{\bullet}{2}0 + \underset{\bullet}{2}0} + \underbrace{9 + 6} = \overbrace{\underset{\bullet}{4}0 + \underset{\bullet}{1}5} = 55$$

Raisa gradually made the transition from the detailed notation to the usual notation, but she still used colored arrows and brackets. This facilitated the correctness of her solutions.

Later, in studying addition and subtraction with three-digit numbers, supplementary lessons were again required. The first examples were solved again by using colored and detailed notation.

In columnar notation, the columns of numbers denoting units, tens, and hundreds were separated from each other by a colored line. A detailed notation was also required.

$$\begin{array}{l} 300 + 80 + 1 \\ 400 + 10 + 7 \\ \hline 700 + 90 + 8 = 798 \end{array}$$

2. Rita B., age 12 1/2. Third grader in the auxiliary school. She is an average pupil in this grade, but her position in arithmetic is very weak. She gets D grades in all her assignments. Despite this, Rita works systematically, obstinately, and persistently. She solves problems in class and at home. She is afraid and does not want to remain in third grade for a second year. She does extra studying willingly.

It cannot be said that Rita knows nothing of arithmetic. On the contrary, if her grades are compared with the scope of her arithmetic concepts and knowledge, the evaluation of her knowledge seems wrong. Rita reads all numbers from 11 to 20 correctly. She knows the table of the sums and differences of numbers up to ten. She knows the multiplication and division table quite well. She understands the essence of each arithmetic operation. Nevertheless, she cannot cope with assignments within the abilities of the rest of the class.

In experimental lessons with the girl, it was established that she could not give a decimal analysis of numbers. This was the basic cause of errors she made in her arithmetic assignments. The direction of Rita's work in solving arithmetic problems is extremely interesting. She is convivial and friendly until she encounters difficulty. Then the girl changes radically. Her voice becomes quieter, she lowers her eyes. We cite an excerpt from one of the lesson records.

Rita was asked to solve a problem on subtracting two numbers: 42 - 20.

Teacher: What must be done?

Pupil: I must solve the problem.

T: What problem?

P: To take 20 from 42.

T: What is the arithmetic operation?

P: 42 take away 20.

T: How many numbers in this problem?

P: Two.

T: What do you have to do?

P: Take 20 from 42.

T: Which of these two numbers is bigger?

P: 42; 20 is smaller.

T: Which number do you take away from, if one is larger and the other is smaller?

P: From 42; it's bigger, and 20 is smaller.

T: How will this be done?

P: [Silence]

T: Tell me how you will take 20 away from 42.

P: [She silently solves the problem, becoming confused, and whispers something to herself.]

T: Please say louder how you are solving it.

P: [She is silent and continues to work on the solution.]
[The solution is correct.]

I remark that the answer is correct. Rita becomes more lively and answers freely and rapidly when asked to tell how she solved the problem.

"From 42 I must take away 20. You take two from four and get two, and from two units you take away zero, and get two. The total is 22."

The second time this same problem is solved differently: "Take two from two, that gives zero, and four remains. The total is 40."

When asked to solve this same problem repeatedly, new incorrect solutions are obtained: "Take away 4 from 20, that gives 16, and we have 2 left. That will be 18." Or: "Take away 4 from 20, it gives 16." She could not go further, "I don't know where to put the 2."

When asked to break down each of the numbers into tens and units, she first refused, saying, "I don't know how." Then she made an attempt (with our guidance): the number 42 is taken first. She counts the tens: 10, 20, 30, 40, and then makes this notation:

$\underline{42 - 20 = 22}$ $\underline{64 - 41 = 23}$

$\cancel{10}, \cancel{10}, 1\underline{0}, 1\underline{0}, \underline{2}$ ~~10 10 10 10~~ $\underline{10}$ $\underline{10}$, $4 - 1 = \underline{3}$

$\cancel{10}, \cancel{10}$

$\underline{57 - 35 = 22}$ $\underline{33 - 12 = 21}$

~~10 10 10~~ $\underline{10\ 10}$, $7 - 5 = \underline{2}$ $\cancel{10}$ $\underline{10\ 10}$; $3 - 2 = \underline{1}$

$\underline{84 - 62 = 22}$

~~10 10 10 10 10 10~~ $\underline{10\ 10}$, $4 - 2 = \underline{2}$

$\underline{100 - 75 = 25}$

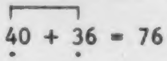
~~10 10 10 10 10 10 10~~ $\underline{10 + 10} + \overset{5}{\cancel{10}} - 5 = \underline{25}$

This step was the transition to solving simple problems on adding and subtracting two-digit numbers. She was continually successful after being asked to designate the numbers showing the quantity of tens with a dot, or to join them with a bracket.

Assignment

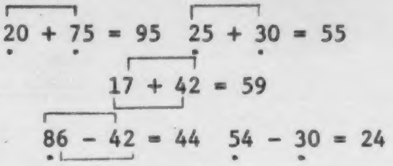
$40 + 36 =$

My illustration:



$$\underset{\cdot}{4}0 + \underset{\cdot}{3}6 = 76$$

Rita's Notation of the Solution of Problems



$$\underset{\cdot}{2}0 + \underset{\cdot}{7}5 = 95 \quad \underset{\cdot}{2}5 + \underset{\cdot}{3}0 = 55$$

$$17 + 42 = 59$$

$$\underset{\cdot}{8}6 - 42 = 44 \quad \underset{\cdot}{5}4 - \underset{\cdot}{3}0 = 24$$

In solving complex problems on addition (without regrouping) the work was done correctly, but at a slower rate. Of twenty cases, none contained any mistakes in solution. These were of the type

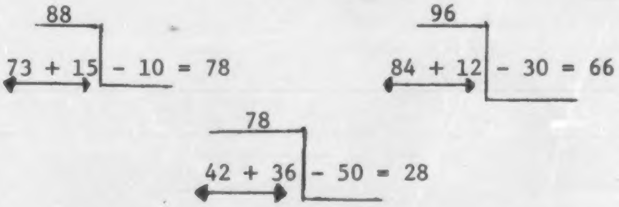
$$\underset{\cdot}{4}3 + \underset{\cdot}{2}1 + \underset{\cdot}{3}5 = 99$$

$$\underset{\cdot}{1}6 + \underset{\cdot}{3}1 + \underset{\cdot}{4}2 = 89$$

$$\underset{\cdot}{3}0 + \underset{\cdot}{2}4 + \underset{\cdot}{4}3 = 97.$$

However, in the solution of complex problems in which addition alternated with subtraction, difficulties again arose. The usual technique of the school methodology (to write out the first result over a line) had been mastered by Rita, but the work went very slowly, since the four elements marked with dots prevented her from breaking down two numbers necessary for the second arithmetic operation. Rita was shown a notation separating the first operation from the second with a line, and this made it possible for her to obtain correct solutions.

A model of the notation separating the problem's parts is:



$$\overset{88}{73 + 15} - 10 = 78 \qquad \overset{96}{84 + 12} - 30 = 66$$

$$\overset{78}{42 + 36} - 50 = 28$$

The girl was also shown the form of sequential linear notation of a solution, which she easily mastered. Subsequent work was very productive.

A model of sequential linear notation of solution is:

$$\underset{\bullet}{41} + \underset{\bullet}{27} - 30 = \underset{\bullet}{68} - \underset{\bullet}{30} = 38$$

The same sequential notation was utilized in the solution of complex examples on multiplication and addition or subtraction, on division and addition or subtraction.

We cite the notations of the solution of problems which, with the help of dots, was fully mastered by Rita:

$$8 \times 8 - 30 = \underset{\bullet}{64} - \underset{\bullet}{30} = 34$$

$$7 \times 6 - 20 = \underset{\bullet}{42} - \underset{\bullet}{20} = 22$$

$$7 \times 7 + 50 = \underset{\bullet}{49} + \underset{\bullet}{50} = 99$$

$$6 \times 6 - 20 = \underset{\bullet}{36} - \underset{\bullet}{20} = 16$$

In completing arithmetic operations including regrouping, the same technique was used with the indispensable aid of detailed notation:

$$\underset{\bullet}{27} + \underset{\bullet}{35} = 50 + 7 + 5 = \underset{\bullet}{50} + \underset{\bullet}{12} = 62$$

This technique was effective because Rita's attention was constantly fixed to the problem being solved. The sequence of the elements of the solution, designated by colored dots, became visual.

We applied this same method in the process of teaching the solution of arithmetic problems to other pupils of the auxiliary school as well. In organizing the exercises, our basic proposition was the necessity of analyzing and combining the work of each pupil.

The concluding stage in the completion of each exercise was the oral account given by each pupil. This was not supposed to form any particular rule for completing an operation, but, supported by the notation of the solution, was to determine the course of the solution of the assignment. Thus, the preliminary plan and the possibility for constant independent checking of the completion of the assignment, and the subsequent oral account, promoted understanding of the solution completed by the pupil. The control dots the pupil placed below the digits or the control arrows with which he designated the necessary elements of the various numbers occasionally had to be replaced by detailed notation. This protected the pupil from the unconscious repetition of the exercises and consolidation of a sluggish stereotype of operation. This was also the aim of changing the exercises, essentially homogeneous, but differing in the numerical components.

An essential and significant element of the indicated study activity was the pupil's interest, which was increased with each visible success--the problems were solved without those continual errors which had previously plagued the pupils' work. The very process of their activity not only became visual, but also was intimately combined with the explanations. This combination of visual and oral means protected the child's thinking from deviations and errors in his completion of the exercises.

After it had been established that calculation could not be the source of errors in solving a problem, we attempted to ascertain the difficulties encountered by the poorer pupils when reproducing the conditions of a problem.

Reproducing the Conditions of an Arithmetic Problem on a Visual Basis

In school practice, reproduction of an arithmetic problem's conditions is the constant element preceding its solution. Most often the teacher asks the pupils to repeat a problem's conditions after he has read or told them to the pupils once or twice, imagining that this repetition serves as a means promoting solution of the given problem. In this the reading of the problem's conditions is not accompanied by notation of the numbers, or illustrated in some form (model, sketch, drawing). In this case the teacher uses verbal means and requires a verbal answer from the pupils. It is supposed that such an assignment is completely within the pupils' capabilities at any stage of their school instruction, for the assignment itself was chosen accordingly (smaller numbers and fewer words in the problem's conditions for the lower grades, increasing them for later years of instruction).

The data of our pedagogical investigation did not confirm the correctness of this widespread technique of instruction. What is correct is the central premise that the solution of an arithmetic problem is influenced by the mastery of the conditions and their reproduction by the pupils. However, this reproduction is necessary and useful only when it is given correctly by the pupils, although not necessarily verbatim. Whenever the conditions are distorted in reproduction, the necessary solution cannot be given. The solution is realized either correctly or incorrectly. This holds true with respect to the solutions of problems by pupils of various grades (grades 2-4 in our data).

This is especially apparent with pupils whose intellectual ability is most limited. We present selected comparative numerical, data (by grades) from the materials of our investigation, illustrating and supporting them with cases from individual lessons conducted with pupils as control investigations.

Grade II (special). 15 pupils in the class. The teacher has special training. There is good discipline in the class. The children work at their lessons systematically and with interest.

On the day before the lesson, we conducted individual lessons with each pupil. The aim of these studies was:

1. To ascertain the pupils' ability to reproduce the conditions of an arithmetic problem.

2. To disclose the interdependence between the reproduction and the solution of an arithmetic problem.

3. To construct a plan for an experimental lesson after analyzing the observations made.

The pupils were given this problem: "Someone bought biscuits and donuts. The biscuits cost 7 rubles, the donuts 4 rubles. How much was the total purchase?"

As a result of the individual lessons, the pupils were required to reproduce the problem's conditions after it had been stated twice. Of the 15 cases, 7 were unsuccessful. In no case was there a solution.

In three cases only the first, introductory, statement was retained: "Someone bought biscuits and donuts," "Someone bought biscuits and bought some donuts," "Biscuits and donuts were bought in a store." Even in these cases no solution to the problem was forthcoming. There was a girl who could reproduce only three words: "Someone bought biscuits." She also did not solve the problem.

In two cases the reproduction retained the first or only a part of the second element and the last element of the conditions: "How much was the purchase?" or "Someone bought biscuits. How much did they cost?" Of course, these reproductions did not give the necessary result--the children could not solve the problem. In reproducing the problem's conditions, one pupil remembered one of the numbers, which caused him to introduce this number in the answer. From his point of view, this was the result--the solution of the problem. "Someone bought biscuits for 7 rubles. How much did it cost?" After he had communicated the conditions in this way, he gave the answer, "7 rubles." This is no solution. Only in one of the 15 cases was an approximate reproduction of the problem's conditions given, resulting in a solution of the problem, which, however, was incorrect.

Reproduction	Solution
Someone bought something for 7 rubles and something else for 4 rubles. How much in all was necessary?	7 - 4 = 3. 3 was necessary.

The teacher's guiding questions did not help the situation. The solution was not changed. Statements like "From 7 one can take 4, giving 3" and "correct, because they solved 7 - 4 = 3" were advanced as grounds for substantiating this solution.

Thus, the overall results are clearly unsatisfactory. It was impossible for any pupil in the class to reproduce the problem's conditions. Each of these pupils had been in school not less than three years and during his arithmetic lessons (with very few exceptions) a teacher has had the pupils repeat the conditions of some problem every day. An ability to reproduce the text of a problem's conditions has not been cultivated. A significant number of the pupils displayed total helplessness when trying to reproduce a problem's conditions--they quit. But even the children who set about completing the given assignment were unable to realize it. In the conditions of problems reproduced by the pupils there were no numbers, necessary for any arithmetic problems. Even in those rare cases when numbers were remembered, the problem was not understood, and the solution did not occur.

Following an analysis of the facts obtained as a result of individual studies, a plan for an experimental lesson was worked out. According to this plan, a lesson was given in two second grades and one third grade. The children of one of the grades were not given individual lessons. In this grade the lesson was begun with a statement of the conditions of the problem and a request that they be repeated. This was not done in the other classes, since we already knew the pupils' capabilities in reproducing a problem.

The curricular-methodological aim of the lesson was to lead the pupils from the solution of a simple arithmetic problem to the solution of a complex problem.

The experimental aim of the lesson was 1) to check the effectiveness of pupils' activity in solving an arithmetic problem when shown the problem's conditions in different forms, verbal and verbal-visual; 2) to disclose the relationship between the reproduction of the conditions and the solution of the arithmetic problem.

Successive course of the lesson

I.

1. Direction of the pupils' attention to the first assignment of the lesson: to listen carefully, try to remember and repeat the problem's conditions.
2. Problem's conditions stated twice.
3. The pupils are questioned. (A good, average, and poor pupil are questioned. The teacher writes out their answers.)
4. Solution of problems (with notation in the notebooks, then on the blackboard).

II.

1. Presentation of the problem's conditions in visual-verbal form.
2. Pupils' reproduction of the problem's conditions using a sketch and the teacher's questions.

3. Reproduction of the problem's conditions using only a sketch.
4. Solution of the problem with notation in the notebooks and then on the class blackboard.

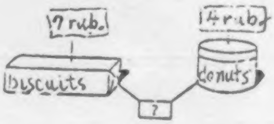
III.

1. Transformation of a simple problem into a complex one (sketching a supplementary element of the problem, analysis, and repetition of the problem's conditions, using a drawing and the teacher's questions).
2. Pupils' reproduction of the problem's conditions, using visually illustrative notation on the blackboard (the elements of this notation are sequentially indicated by the teacher).
3. Reproduction of the problem's conditions using a notation made on the class blackboard (without the teacher's indicating the individual elements).
4. Solution of the problem and subsequent notation in the children's notebooks and on the blackboard.

IV.

Supplement to the lesson plan (or its fourth part). In case both parts of the problem are solved and the outlined plan is fulfilled, another complex problem is offered for solution. In this problem the objects purchased and their cost remain as before, as does the sum of the money involved. The problem is expanded by the addition of a new element--another object is purchased. Otherwise this part IV is used as material for a subsequent lesson.

Problems and Models of Their Notation on the Blackboard

Texts	Forms of the Notation of the Conditions and Solutions
Problem 1 (simple) Someone bought biscuits and donuts. The biscuits cost 7 rubles, the donuts 4 rubles. How much was the whole purchase? 7 rubles + 4 rubles = 11 rubles The purchase was for 11 rubles.	 Figure 6

Problem 2 (complex)

Someone bought biscuits and donuts. The biscuits cost 7 rubles, the donuts 4 rubles. He gave the cashier 20 rubles.

How much change did he receive?

1) 7 rubles + 4 rubles = 11 rubles

2) 20 rubles - 11 rubles = 9 rubles

He received 9 rubles in change.

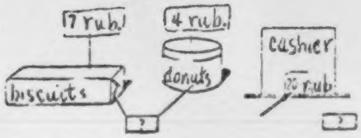


Figure 7

Problem 3 (complex)

Someone bought biscuits, donuts, and candy. He paid 20 rubles for the whole purchase. The biscuits cost 7 rubles, the donuts 4 rubles.

How much did the candy cost?

1) 7 rubles + 4 rubles = 11 rubles

2) 20 rubles - 11 rubles = 9 rubles

The candy cost 9 rubles.

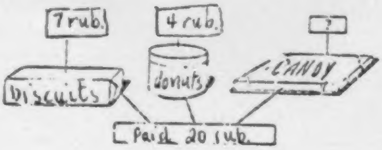


Figure 8

The problems are given in succession, one after another. The teacher relates the conditions and simultaneously draws on the blackboard, always addressing the class.

What was bought? (Biscuits.) A box of biscuits are drawn.

And what else? (Donuts.) A box of donuts is drawn.

So, what was bought? (Biscuits and donuts were bought.)

And what was paid? (Money)

How much were the biscuits? (The biscuits cost 7 rubles.) The cost of the biscuits is written down.

How much were the donuts? (The donuts cost 4 rubles.) The cost of the donuts is written down.

Now let's look at the drawing and tell what was bought and how much money was paid.

The teacher indicates each element. Pupils answer individually (two or three pupils of varying arithmetical ability answer).

What is the question of the problem? What is asked in the problem?

The teacher collects several answers to this question and obtains the one best in form ("How much did the whole purchase cost?").

Having selected the best answer, the teacher asks the pupils to establish the number of words in the problem's question. There are 7. With the teacher, the children recite the question word for word and tap their hands on the table for each word: first, "how," second, "much," etc. In the rectangle where the question of the problem should be written a pupil draws seven vertical lines and puts a question mark. The teacher asks the pupils to repeat the problem's question (two or three times), and then relate the whole problem. The teacher indicates parts of the drawing, and the children reproduce the text of the problem's conditions. Then solution of the problem is proposed. The pupils tell the teacher the results of their solutions, indicating the arithmetical notation or only the final part of it (the answer), which each pupil made in his own notebook. After all the children's work has been examined, the solution of the problem is written by one of the pupils on the blackboard. Two pupils are called up; one relates the conditions and then the solution of the problem; the other accompanies the narration by indicating the parts of the problem.

The other problems are analyzed in approximately the same way.

Following the lesson the pupils are questioned individually in order to ascertain what was mastered at the lesson. (Verbal reproduction of the first and each subsequent problem was required. When this was unsuccessful, a drawing was shown.)

The results of reproduction of the conditions, and depending on this, of the solution of the problem, were strikingly different from those obtained the first time, i.e., after the problem had been given only verbally.

Most importantly, not one pupil gave up. In reproducing the first problem, two pupils omitted numbers (both numbers in one case, and the second number in the other case). In the solution the numbers were not named, but a final result was given. A request to write out the solution was complied with, and the lines of the notation were precise. In reproduction of the second problem, the results were somewhat inferior, but incomparably better than what we obtained the first time. In these reproductions there were the following flaws:

1. In 4 cases the sum of money given the cashier was changed (to 200, 10, 29, and 27 rubles).

2. In 7 cases the word "change" was not used. Here the problem's question was formulated thus:

How much money did he obtain? (Two answers.)

How much did the cashier return? (Three answers.)

How much did he have left?

How much more could be bought?

In all these cases one may speak of imprecise formulations, but not of logical mistakes in the question.

The solution of problems was incorrect in five cases, but in only one of those cases in which the total sum of money had been changed. The sum of 200 rubles, named by one in the conditions, did not figure in his arithmetic operations. Here all numbers corresponded precisely to the problem's conditions. The sums of 29 and 27 rubles were not used either. These numbers were included in the problem's conditions, but the sum of 20 rubles were given in the solutions. Thus, the solutions did not reflect the changes brought into the conditions of the problem when they were repeated. A pupil included the sum of 10 rubles in the solution too. The problem in this case incorrectly solved in its second part. The pupil wrote: 10 rubles - 11 rubles = 9 rubles. But when his attention was called to this mistake, he did not correct the whole solution, but merely crossed out the 9 and put a 1 in its place. In his first solution the 9 had been taken from the problem's final answer and placed to the right of the equals sign on the arithmetic line. The unit was obtained by subtracting the minuend from the subtrahend, since the latter was greater than the former. He did not return to this solution after he related the problem's conditions using a drawing--he then solved the problem correctly. In two solutions of a single problem, the pupil did not see the total and that alteration which he himself had introduced. A correlation of the two solutions was impossible. There were no connections between the first and second assignment; each of them appeared independently.

Mistakes in all other solutions (four cases) occurred in the second part of the problem and consisted of the following:

1. 11 rubles were subtracted from 20 kopeks, resulting in 9 rubles.[*]
2. Notation of the numbers was made without designation of the units in the answer: 9 kopeks.
3. In subtracting 11 rubles from 20 rubles, one girl obtained 7 rubles.
4. A reverse sequence of numbers in the subtraction was given: 11 rubles - 20 rubles = 9 rubles. The number obtained in solving the first part was taken as the starting point of the second part.

Reproduction of the third problem's conditions (this problem was solved at the lesson), as we ascertained from the general questioning, was very difficult for most of the pupils in the class. Only two of the 15 pupils gave more or less verbatim texts of the conditions and complete solutions.

> Someone bought biscuits and donuts. The biscuits cost 7 rubles, and the donuts 4 rubles. The person gave the cashier 20 rubles. How much did the candy cost?

Retained in this case were those elements which were repeated several times in the solutions of the first two problems, and which were constituent parts of the third problem's conditions, although in a somewhat different sequence.

The second pupil's reproduction of the text was:

> Someone bought biscuits and donuts. The biscuits cost 7 rubles, and the donuts 4. He gave 20 rubles. How much did the candy cost? He gave the cashier 20 rubles and received no change. How much did all the candy cost?

Here too one can see the same regularity, i.e., the elements of the problem which were constituent parts of all the previous problems were not forgotten. The problem's question was given twice, since after the supplements to the initial given conditions of the problem, the text of the problem was unusual and incomplete. The ending in the form of a question made the conditions usual, and the transition to the solution was more substantiated.

---

[*] One ruble equals 100 kopeks (Trans.).

In two cases the text of this third problem's conditions was composed of elements of the first two with only one word, "candy," added:

> Someone bought biscuits, donuts, and candy. The biscuits cost 7 rubles, the donuts 4 rubles, and the candy. He gave the cashier 20 rubles and obtained change. How much change did he receive?

> Someone bought biscuits, donuts, and candy. The biscuits cost 7 rubles, the donuts 4 rubles. He paid 20 rubles. How much change did he receive?

The solutions of these problems were identical to those of the second problem. The cost of the candy was not determined, although candy was entered into the number of purchases.

In the remaining cases the children either did not give the conditions of the third problem (they "forgot") or gave only the first sentence or the first sentence and the question of the problem. Only when asked the supplementary question "Were there numbers in the problem?" did each of the children name all three numbers. But even after this, the problem's conditions were not constructed. The solution was replaced by the final answer of the second or third problem: "9 rubles change," "9 rubles should be the change" (in seven cases); "The candy cost 9 rubles," "He paid 9 rubles for the candy" (in four cases).

It is too difficult for the children to discern the fine distinction in the elements of the latter two problems, since the children are not yet accustomed to juxtaposing and comparing texts of conditions and solutions of arithmetic problems. Only in later lessons were they able to obtain the necessary results.

The following conditions were decisive in the successes achieved: first, the possibility of help from pictures of the objects purchased, from the visually perceptible correlation of the cost of each of them, from the clearly and visually designated question of the problem; second, the repeated review of the individual parts of the problem's conditions, once again accompanied by visible unification of these parts into an integral unit.

In the general questioning, each pupil was oriented so that he would consciously come to the problem's question, either by watching a demonstration, or by indicating the elements of the conditions himself. This demonstration produced goal-oriented activity according to a clear plan. The text of the problem's conditions, before it was reproduced as a whole, was composed of parts, each of which, when singled out, was understood by the pupil. The visual plan for his analytic and synthetic mental activity protected the child's speech against accidental introduction of extraneous material and digressions from the main course.

First we should note a decrease in the number of pupils' refusals to reproduce the problem, for reasons like "I forgot," "I don't remember," "I can't" in grades two and three. Second graders of the special (weaker) class gave up in 33.3% of the cases, regular second-graders gave up in 23.5% of the cases, and third graders, 16.6% of the cases.

In an examination of the texts of the problem's conditions which had been reproduced by the pupils, some definite order was disclosed: the first part of the problem's conditions, which contained the names of the objects purchased (i.e., the five words "Someone bought biscuits and donuts"), was expressed best in the reproductions of the pupils of all classes. The last part (the problem's question, "How much did the whole purchase cost?") was significantly less well expressed, and the most poorly expressed was the main part of the conditions, containing the data of the cost of the objects purchased (the biscuits cost 7 rubles, the donuts 4 rubles).

We present a summary table of the results of the reproduction of each part of the problem (See Table 1).

Although these results do reflect a positive influence from the instruction, they basically illustrate the poor potential of the mentally retarded schoolchildren for understanding the conditions of the arithmetic problem presented in purely verbal form. Even some of the third graders, who had already had some experience in solving arithmetic problems, did not know that the problem must necessarily have numerical data. Of 50 pupils in all three grades, 14 persons

TABLE 1

PERCENT OF PUPILS IN EACH GRADE WHO REPRODUCED PARTS OF THE PROBLEM

Grade	Parts of the Text		
	Someone bought biscuits and donuts.	The biscuits cost 7 rubles, the donuts 4 rubles	How much did the whole purchase cost?
2 (special)	40.0	5.7	13.3
2	66.1	24.4	35.3
3	73.3	31.7	45.8

(28%) failed to give both numbers in their reproduction of the conditions, and in one instance only one number was named. The larger part of all cases occurred in the special second grade (one and one half times greater than in each of the other two grades).

The lessons we developed on the basis of data obtained as a result of individual studies conducted in the second and third grades of several auxiliary schools (in Moscow, Riga, Noginsk) confirm that children reproduce the texts of arithmetic problems much more successfully when they have the opportunity, during the reproduction process, of relying on the basic visual elements of the text, corresponding to the teacher's speech. This occurred whenever we presented new study material or a new, still familiar arithmetic problem whose solution required collecting all one's previous experience in order to select from it everything useful for understanding and solving the problem.

We gave the third and fourth graders a simple arithmetic problem unlike those they had solved in school, but one which was freely and easily solved by second and third graders of the general public school. After the teacher had read the text of the problem twice, the children were asked to solve the problem. No correct solutions resulted. The children's attempts to reproduce the text of the conditions, which were necessary for solving the problem, did not give the result required and did not foster completion of the assignment.

Text of the Problem

> It is 10 kilometers from a kolkhoz to a forest, and from the kolkhoz to a mill on the same road it is 4 kilometers. How many kilometers from the mill to the forest?

In the problem's text there are 34 words [22 words in the Russian]. Not one pupil reproduced the text of the conditions precisely and completely. There were occasional approximate reproductions in the third grade. This increased somewhat in the fourth grade.

> It is 10 kilometers from a kolkhoz to a forest, and to a mill it's 4 kilometers less. How much to the forest from the mill? 10 kilometers - 4 kilometers = 6 kilometers. To the mill it's 4 kilometers and to the forest, 6. (Third grade)

> It is 10 kilometers from a kolkhoz to a forest, and to a mill on the same road from the kolkhoz it is 4 kilometers. How many kilometers from the mill to the kolkhoz? No. To the forest? 10 kilometers - 4 kilometers = 6 kilometers. Because 4 kilometers to the mill are already travelled, and we still don't know how many to the forest, but there were 10, so we take away. (Fourth grade)

These are the best reproductions and solutions. A significant number of third graders understood the words "on the same road" as an additional route.

> It is 10 kilometers from a kolkhoz to a forest and on the same road 4 kilometers to a mill. How many kilometers altogether to the forest?
>
> 10 kilometers + 4 = 14 kilometers.

> He has to go 10 kilometers from the kolkhoz to the forest, and 4 more kilometers to a mill on the same road. How far altogether does he have to go?
>
> 10 + 4 = 14 kilometers.

> It is 10 kilometers from a kolkhoz to a forest, and to a mill it is 4 kilometers. He has to go along the same road so as not to get lost. 14 kilometers in all. He has to go far--10 kilometers and another 4 kilometers. It would be 10 kilometers from the forest to the kolkhoz.

The fourth graders made this same mistake, although less frequently.

In reproducing the conditions, many pupils changed the problem's question. Its formulation often did not coincide with the conditions.

Third graders said: "How far must one travel in all?" "How far did the kolkhozniks go into the forest?" "How far in all to the mill?"

Of 15 third graders, six did not give a question at all. The text of the conditions is transformed into the solution. All these pupils had incorrect answers.

In the fourth grade the conditions were not completed by the question in four of the 15 cases. An incorrect solution resulted in all these cases. "How far remained to the mill?" "How far to the forest?"

On a positive note; in all the reproductions of the conditions, the numbers were remembered. And when the text of the problem's conditions was accompanied by a graphic demonstration, the results of both the reproduction and the solution were noticeably better. The weakest pupils were drawn to the sequential demonstration (analysis) of the drawing's basic elements.

The child's speech became more sure and confusion disappeared. The task was completed with a correct solution. Four third graders and three fourth graders required a second reproduction of the conditions using a drawing before each could go on to the solution (Figure 9). The solutions were correct.

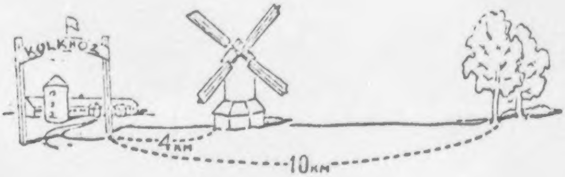


Figure 9

In the experimental lessons and the individual studies we made with the children of the different classes, we were able to determine the necessity of concrete objects or drawings of the conditions or an arithmetic problem for its reproduction and subsequent solution.

It is interesting that the ability to single out the essential and main features of each arithmetic problem in the practice of school instruction is cultivated very slowly. The data we obtained from our experimental lessons in the fourth grades of several auxiliary schools (in Moscow, Leningrad, Noginsk) also confirm this.

We present here some excerpts from records of these lessons.

On the class blackboard are presented incomplete conditions from two identical arithmetic problems (without the questions); several numbers determining the cost of objects purchased are missing. The pupils' attention is called to the conditions of the first problem.

1. They bought 4 kg of sugar at ___ rubles for 1 kg and 4 kg of granulated sugar at ___ rubles for 1 kg.

2. They bought 4 kg of sugar at ___ rubles for 1 kg and 4 kg of granulated sugar at ___ rubles for 1 kg.

The teacher asks the pupils to read these conditions and answer the question: "Can the problem be solved?" Of 10 classes in which this lesson was conducted, there was not a single one in which the pupils fully recognized the impossibility of solving the problem. In all classes the opinions were divided (with small vacillations in either direction). A significant number of children considered the solution possible. And only after asked to begin solving the problem, after the children had picked up their pens and were ready to begin writing, did doubts arise. "It can't be solved, there are no numbers," "You can't solve it because we don't know how much the sugar costs," "It's not known how much the sugar costs." In those classes in which the word one was placed before the word kilogram in these conditions, the answers were more precise: "It can't be solved; it's not said how much one kilogram of sugar costs" (the same for the granulated sugar).

When the necessary numbers were inserted, the children again were asked: "Can the problem be solved now?" And again there were quite a few positive answers. We cannot help noticing that these answers were given less hastily than in the first case. The suggestion to begin the solution, given only to those pupils who had stated that a solution was possible, evoked vacillation. These vacillations occurred in both groups of children who began the solution very amicably. Some who had just stated that the problem could be solved did not begin working, while others who had spoken of the impossibility of any solution leaned over their desks and began working. After a while those solving the problem stopped working and announced that a solution was impossible. "There is no question," "It is not known what is asked," "The problem's question has to be asked." A question was added without trouble: "How much did the whole purchase cost?" There were no incorrect formulations of the problem, although in individual cases the formulations were not so smooth: "How much do they have to pay for all the purchases?" "How much did they spend for sugar and granulated sugar?" After a question had been stated, a solution of the problem resulted in all cases.

The children's attention was switched to reading the text of the second problem. Numbers were inserted--the same numbers which had been appropriately substantiated before: "We have identical prices everywhere," "If the prices are decreased, they are decreased all over the country." Several pupils even knew that "we have state prices."[2]

Immediately all pupils, without exception, announced that the question of the problem could not be answered. However, statement of the question was difficult. Almost all pupils in all classes offered the same question with which they had solved the first problem. And when this question was not accepted, since a problem with the same question had just been solved, many pupils were discouraged: "The problem is the same," "They are identical," etc. The request to ascribe some other question to the problem put the pupils in a difficult position. Again and again the pupils read the conditions of the problem and saw no essential distinction in the two parts of the problem: identical quantities (4 kg) and different prices of the same unit of weight (kg) of sugar and granulated sugar. The children were unable to compare these prices and ask "How much more expensive is 4 kg of sugar than 4 kg of granulated sugar?" or "How much more expensive is the amount of sugar bought than the amount of granulated sugar bought?" Their thought followed another direction; they could not detach themselves from the question they had just formulated. They

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[2] The lessons were conducted shortly after the value of currency had been increased, and the children knew about this from conversations at home and in class; many were buyers themselves.

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repeatedly stated the question they had used for the first problem, altering its formulation. The children offered 14 variants of one and the same question. These variants were similar in essence and different only in their external formulation.

"How much money was spent for the purchase?" "How much did they pay for all, the sugar and the granulated sugar?" "How much did they give the cashier?" "How much must be spent for everything?" Whenever they were asked to think about the difference between this question and that of the first problem, the children found it in the fact that there were seven words in the first one and more here, that is was the whole purchase there, and only sugar here, etc. Only after they had been asked "How do all your questions differ from the question of the first problem, by words or by meaning?" did the children cease their unsuccessful attempts to alter the form of the question, but they asked their own question, "How should it be?" They were asked to look carefully at the problem's conditions. The first pair of data (the quantity and cost of 1 kg of sugar) was underlined in chalk of one color, and the second pair, in chalk of another color. The children were asked to compare and answer the question: "What is identical and what is different in the underlined parts?"

In four of the ten classes there were children whose comparisons followed the line of juxtaposition by form--"in the first you have kg written big and in the second not so big," "there you wrote the 4 a little differently," "you underlined the first in blue and the second in yellow," etc. Only after several questions had been asked, which were accompanied by a search for the notation of the necessary elements in the text, were the necessary answers forthcoming.

How much sugar did they buy? Indicate it and state it.

How much granulated sugar did they buy? Indicate it and state it.

They bought 4 kg of sugar and 4 kg of granulated sugar (teacher indicates). What can be said about the weight of each of these?

The weight of the sugar and the weight of the granulated sugar are identical.

How much did they pay for 1 kg of sugar? Indicate it and state it.

How much did they pay for 1 kg of granulated sugar? Indicate it and state it.

Let us compare the numbers denoting how much 1 kg of sugar and 1 kg of granulated sugar cost. What can we see in comparing these numbers? (Point out the numbers.)

The numbers are different. The numbers are not identical.

The weight of the sugar and that of the granulated sugar bought are identical, but the prices are different.

What can we find out from these data?

Only after this was the original and determining necessary question of the problem obtained: "How much more expensive is the amount of sugar bought than the amount of granulated sugar?"

After the second problem was solved the solutions of both problems were analyzed, and the pupils were asked to compare the conditions and the solution. What is identical and what is different in the conditions of the problems? This encountered no special difficulties. Correct answers were also given to the questions "Why are the problems solved differently although everything is identical except the question?" and "Can you solve a problem if you haven't read and understood its question?"

These examples suffice to convince us how poorly the auxiliary school children analyze, differentiate, and generalize. The means of instruction are incorrect; the teacher does not organize special conditions necessitating the pupil's independent isolation of the problem's main elements, and his independent formulations and conclusions, based on visual means of instruction.

Conclusions

The major shortcoming in teaching arithmetic problem solving to mentally retarded schoolchildren is the teachers' inability to create conditions necessary for the pupils' goal-oriented and organized cognitive activity. The teacher frequently assumes functions quite different from those he is called to realize. At arithmetic lessons the teacher frequently does not act as an organizer of conditions in which he guides the child's speech and thought along the road to active and more independent activity, protecting him against slips and digressions from the outlined plan of work. He takes on himself all the activity of operation: he does not lead the pupils to conclusions, but himself states them; he does not wait until the children, operating according to the proposed system of exercises, are able to formulate one or another rule or generalization, but gives this rule in prepared form; he does not contemplate forms of combining visual and verbal means of instruction in accordance with the aims of each lesson, but uses them in the course of the lesson or holds to established tradition, which is often insufficiently checked and substantiated. In all these cases the teacher does not create the conditions necessary for the formation and consolidation of the generalizing connections which would promote the singling out of basic features, in problem solving, from much more numerous secondary features, with the aim of developing the analytic-synthetic abilities of mentally retarded schoolchildren. The incomplete forms of this kind of a teachers' work appear especially clearly in classes in which the pupils' intellectual abilities are greatly limited.

Productive activity in solving arithmetic problems becomes possible for the mentally retarded children only as a result of lengthy, systematic, special instruction. The necessity of such instruction is conditioned primarily by those peculiarities characteristic of all cognitive activity and, as a part of it, of the mentally retarded child's scholastic activity.

When arithmetic problems are being solved one can frequently see clearly expressed dissociations in the activity of the signal systems; one may continaully observe that the formation of new connections is hindered, that very precise analysis is incomplete and even impossible without special educational devices.

In lessons in the classes of the auxiliary school there is a very real need to bring the operation of the signal systems into agreement, thereby promoting a more complete path for the development of speech and thought and, as a result of this, more complete knowledge. An analysis of the problems of developing the mentally retarded schoolchild's speech and thought always leads the teacher to the problem of the correlation of visual and verbal instructional devices.

Arithmetic instruction is no exception, since in teaching arithmetic, and hence in teaching arithmetic problem solving, every assignment may be revealed to the pupils in one of two forms: visual-verbal or purely verbal.

At various stages of school instruction, this main problem may be examined with consideration for, and analysis of, the pupils' experience, since the child's previous experience is always reflected to some extent during and in the result of completion of the assignment, i.e., the solution of the problem.

Whenever the pupil's previous experience is too narrow and consolidated in homogeneous exercises, the solution of new problems is hindered. The solution of a new problem becomes possible more often when, besides verbal instruction, reference is made to some concrete activity in the past, to those visual aids and concrete forms of working with them that accompanied this activity; then the pupil is able to orient himself according to previous experience.

The pupils of the lower grades of the auxiliary school often transfer their past experience unchanged to the assignments new to them. They go about finding ways to solve the new assignment by basing themselves on known visual aids, references to past work with some particular visual aids. The upper-grade pupils also need the support of their previous experience. However, they do not always orient themselves according to concrete models, but according to models of those arithmetic notations which have become customary for them.

The structures of habituated associations, created because of frequent monotonous repetition, often dominate and interfere with the transition to new methods of solving new problems. There can be no independent activity from the children, even if they rely on known visual means, without the teacher's appropriate, guiding, and organizing verbal instructions.

Employment of visual methods in teaching arithmetic to auxiliary school pupils is indispensable, but only when one or another aid is unavoidable, *when work with it makes it possible to disclose the essence of some particular arithmetic relationship, the essence of the question to be answered*.

In organizing the study activity (in our case, teaching how to solve arithmetic problems) the teacher must protect the children especially carefully against erroneous paths in the solution of problems.

In organizing the study activity when using visual aids, the teacher must give attentive guidance until the pupil can consciously use these aids independently. In teaching children to use visual aids, the teacher's work, based on personal experience, is of prime importance. The teacher's personal demonstration and his speech anticipate and organize the pupil's demonstration.

The teacher must follow how the child began his work with the aid, after the teacher gives him guiding instructions, how he completes the first one or two solutions, and finally, the teacher must listen to the child's narration (his verbal account) of how he works with the aid.

One will use for aids, during the first meetings of the instruction, primarily objects; one must consider the selection of these aids very carefully, taking into account the concrete problems which will be solved, and accurately selecting those aids which promote purposeful cognitive activity on the part of the pupils.

(In solving problems involving a purchase the children should have money, or tokens in place of money, at their disposal; in solving problems on measure they should have meter sticks; in solving problems on computing time they should have clock faces, etc.)

In teaching arithmetic problem solving the concept visual means should not be taken too narrowly. Not only objects of counting, not only concrete units of measurement, but all concrete activity in counting and measuring enter into this concept. The forms of notation—illustrative, graphic, structural, a problem's conditions written in colored pencil, colored symbols showing the connection between numbers in an arithmetic problem—are all visual means, and should find wide application in a system of exercises for instruction in how to solve arithmetic problems.

In view of the weak abilities of mentally retarded children for analytic-synthetic activity, many of them, for a long time (from our data—until the fifth grade), are not aware of the function of the question of the arithmetic problem. A system of exercises aimed at revealing the question's function must be organized. At the initial stage of instruction these exercises must be conducted in direct connection with visual means of instruction (tables, models of visual notation on the class blackboard, pupils' independent activity in choosing a question for some particular concrete problem, etc.). At these lessons the teacher's verbal instruction plays the leading role.

Reproduction of the conditions of an arithmetic problem is necessary only if it is correct (if it retains the text), although not necessarily verbatim. The ability to do this is cultivated by exercises. The first exercises are those in which the mentally retarded schoolchild relies on visual aids (a model of the problem, designation of the problem's conditions, etc.). It becomes possible to make advances in the construction of models, drawings, and sketches as the pupil, during the work, is taught to address himself to the text of the problem's conditions. The visual accumulation from one element to another promotes synthesis. Reproduction of the text of a problem is anticipated by continual repetitions of the parts of the whole, from first to last. Direct active participation with objects, clay, sketching, or drawing is a firm basis for the child's speech. Here reproduction of the problem's conditions is realized as an oral account of the concrete visual activity. Hence the quality of this account is much higher and more complete than reproduction following something heard or only read. In instruction in solving arithmetic problems the conditions of each problem are usually given in a prepared form. Often they are not even spoken by the students, but merely read by the teacher (and by the pupil in the upper grades).

This established device, having become almost a tradition for all levels of school instruction, was not supported by our investigation. It must be changed. In introductory exercises one must provide forms of work in which the arithmetic problem is presented in some form of the pupils' concrete activity, and its solution is included as the requisite final stage of this activity. For the first few times this activity is with objects. At subsequent stages in school instruction the arithmetic problem arises once again during concrete work (weighing, measuring, work with clocks and calendars, composing simple diagrams, in the preliminary work in handicraft, etc.) and its solution is once again the requisite final step of the activity. Direct work with objects, units of measure, the pupils' illustrative and graphic work, i.e., the widely employed visual techniques, with organized intelligent verbal guidance on the part of the teacher, make for successful instruction of mentally retarded schoolchildren in solving arithmetic problems.

When studying at lessons of all school disciplines, the pupil is faced with the task of having to solve some kind of problem. These problems are unique, and this uniqueness depends on the characteristics of each of the disciplines.

At Russian lessons there is an orderly system of problems which are consistently included in exercises and prepare the pupils for independent activity in writing, reading, and speaking. At geography lessons the pupils solve problems which combine to provide them cognitive work with the geography map. At natural science lessons the pupils acquire abilities and skills in observing phenomena of the surrounding world. At arithmetic lessons the schoolchild also is taught how to solve problems. These problems are unique and distinct from those he solves at lessons in other disciplines. In the arithmetic problem the pupil encounters questions such as "How much results?" "By how much or how many times is one quantity more or less than another?" In the arithmetic problem the pupil deals with quantities, numbers. In solving an arithmetic problem the child must know that any problem can be solved only if there are at least two numbers in it, and that the solution of a problem always depends on its question.

The mentally retarded child does not learn all of this immediately; a system of special exercises is needed. Each element of this system is worked out by the teacher, considering, first, the continuity between one element and the next and, second, the final concluding goal of the instruction in solving problems. The aim of this instruction is to foster knowledge and skills of independent work in solving practical problems. Arithmetic problems which the pupils of the auxiliary school must solve in their socially useful activity or in serving themselves and people they know in everyday life will rarely occur in the prepared, standard forms which fill the pages of school textbooks. The ability to solve an arithmetic problem from the page of a textbook is more the final phase than the initial phase of scholastic activity. Even solving a simple arithmetic problem requires much knowledge and many abilities that can and should be acquired by the pupils in preliminary exercises.

In defining the sequence (system) of these exercises, the most pressing problem is that of the correlation of visual and verbal means of instruction. At arithmetic lessons the auxiliary school pupils' activity is organized as practical visual-verbal activity. Depending on this, the teacher's role is modified; the teacher is required not so much to instruct by the work as to skillfully organize the pupils' active cognitive study activity.

Formalism in teaching how to solve arithmetic problems can be eliminated only if the established system of school practice is reexamined. The main and decisive factor here is the resolution of the question of combining visual and verbal means in instruction. We consider this investigation the first of the pedagogical investigations directed toward developing and resolving that question.

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SOME FEATURES OF ELEMENTARY ARITHMETIC INSTRUCTION FOR AUXILIARY SCHOOL PUPILS

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It is well known that visual instruction plays a significant role in the development of children's thinking. This role of visual aid in the development of thinking was accurately defined by Ushinskii:

> The object standing before the pupil's eyes or strongly impressed in his memory, by itself, without the means of the alien word, awakens in the pupil a thought, corrects it if it is mistaken, supplements it if it is not complete, leads it into a natural (that is, correct) system if it is illogically placed. For the first exercises it is necessary that the subject be directly reflected in the soul of the child, and, so to speak, in the eyes of the teacher; under his guidance the child's sensations will become concepts, from the concepts an idea will be formed, and the idea will be invested in a word [1:6-7].

The specific character of arithmetic as a school subject allows broad possibilities for developing thinking on the basis of visual instruction. Mastery of arithmetic knowledge necessarily involves comparing, generalizing, etc. Through visual comparison at arithmetic lessons, numerical concepts are formed and calculation develops.

As early as the first stages of the study of arithmetic, when the children are taught to perform the primary arithmetic operations, visual presentation aids the development of their logical thinking. In the practice of teaching arithmetic to mentally retarded children, however, this basic didactic principle is not always realized.

The facts we have observed show a need for the development of the problems of visual presentation, with the aim of forming in children accurate numerical conceptions, with which they can master the whole system of teaching established by the school program.

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Among the teachers of auxiliary schools there are many master teachers. In our investigation we have used the experience of these fine teachers.

Knowledge of Spatial and Quantitative Relationships in Normal and Mentally Retarded Children

The study material in arithmetic for the first grade of the auxiliary school was determined by the program of the Ministry of Education. In the beginning of the curriculum the necessity of an introduction to the course is indicated.

The concepts equal, greater than, and less than have great significance in mathematics. Mentally retarded children must be thoroughly acquainted with these concepts as well as their diverse variants (longer-shorter, wider-narrower, etc.), since these concepts are included in many arithmetic problems. The children must be taught to equate and compare not only units of measurement but also of quantity (greater-less).

The inclusion into the curriculum of an introductory course was made necessary by the insufficient preparation of the pupils for mastery of the study material in mathematics. Hence we considered it necessary first to ascertain the degree of preparedness of pupils of the first grade to master knowledge of an elementary arithmetic course.

We conducted work in the first grades of the auxiliary schools of Leningrad, mainly in Auxiliary School No. 1, and in the upper groups of public kindergartens No. 20 and No. 47 of Leningrad's Dzerzhinskii district.

In the beginning of one school year, we acquainted the teachers with our methodology; and in the beginning of the subsequent school year, the first grade teachers themselves were already conducting the work according to this methodology.

Following this same methodology we conducted work with children of the upper groups of kindergarten, from February through April. In all, 83 children were investigated; 68 of them were mentally retarded, 15 were normally developing children.

The ages of the mentally retarded children were from eight to eleven years (5 eight year olds, 22 nine year olds, 30 ten year olds, 11 eleven year olds). Seven year old children were taken from the public kindergartens.

Of the mentally retarded children, 51 had studied one or two years in mass schools, 11 had entered the auxiliary school from kindergarten, and 6 had entered first grade of the auxiliary school directly from home.

In our investigation we ascertained:

1) the knowledge of spatial features of objects (big-small, thick-thin, long-short, etc., and also greater-less, thicker-thinner, etc., the same in size, the same in thickness, etc.),

2) the knowledge of elementary quantitative relationships applicable to a group of objects (greater-less, and the same in quantity),

3) the ability to count to ten,

4) the ability to perform arithmetic operations and the ability to solve problems with numbers from one to ten.

We examined and constructed the pedagogical experiment as a teaching experiment, in which the experimenter's oral instructions guided and directed the child's action with visual aids. The quality of the child's performance with assignments depended on the quality of the verbal instruction; hence we strove for precision and clarity of instruction.

Moreover, we carefully registered and strictly considered the child's speech accompanying some of his actions or the teacher's actions. We considered the influence of the child's speech on the character of his fulfillment of the assignment, and his correlation of words with actions. When they did not correlate (due to inaccuracy of speech or incorrectness of action),we pointed out the mistake to the child and corrected his speech and actions.

Verbal instructions and help from the teacher directed the pupil's apperception and thinking to the primary, essential aspects of objects and helped divert them from non-essential aspects.

Knowledge of Spatial Features.

Features of objects big-small were explained with a set of ten blocks with edges of one centimeter to ten centimeters. The pupils were asked to give the experimenter the smallest block, then the smallest of the remaining blocks, etc. When the last block had been given away, the set was again placed before the child, and the experimenter asked for the biggest block, then the biggest of the remaining blocks, etc. The whole set was used in this manner.

The features thick-thin were explained using a set of pencils of various thicknesses but of identical length and color. The pupils were asked to give the experimenter the thinnest pencil, then the thinnest of the remaining ones, etc. After this the pencils were placed again before the pupil and he was asked to give the experimenter the thickest pencil, then the thickest of the remaining ones, etc.

A set of ten rulers of equal length, varying from one centimeter to ten centimeters wide, were used for evaluating the pupils' knowledge of the features wide-narrow. The children were asked to give the experimenter the narrowest ruler, then the narrowest of the remaining rulers, etc.; then the children were asked to give the widest, then the widest of the remaining ones, etc. The pupils' knowledge of the features long-short was evaluated in this same way, using a set of sticks of various lengths (the so-called counting ladder).

For evaluating the pupils' knowledge of the features high and low, the counting ladder placed vertically was used again. The pupils were required to give the experimenter the lowest stick, then the lowest of the remaining sticks, etc. Then they were required to give the highest stick, the highest of the remaining ones, etc.

In all cases, we moved on to the next experiment only after using all elements of a set.

The investigation showed that seven year old children from the upper groups of the mass kindergarten could cope easily with all the indicated tasks, while the mentally retarded children fulfilled only 60% of the problems proposed to them.

Of the 30 pupils from the auxiliary school, despite their age of 10 and 11 years, only six could handle fully all their assignments.

Eighteen pupils only partially fulfilled the assignment; six pupils could not cope with a single task.

A comparison of the fulfillment of separate tasks showed that the children coped much more easily with the selection of the largest block when there were blocks of 4, 3, 2 and 1 centimeters remaining, and had significantly greater difficulties when doing the same problem with blocks of 7, 8, 9 and 10 centimeters. This is because the difference between blocks in the first case is more obvious than in the second. For the normally developing children, this circumstance had no significance.

The normally developing children approached the fulfillment of the task immediately after receiving instructions. The instructions did not need to be repeated.

The mentally retarded children after a first explanation of the task could not grasp its meaning. Each assignment had to be repeated two or three times before the child understood what must be done and could begin to do it.

As a rule, children from the kindergarten sought the required objects actively, compared them, for example, placing the bars side by side. They already had a command of this method of comparison. The children checked all the bars in the set (to see if there was a smaller one). "This one, I think, is the tiniest," one of the children of the kindergarten explained after his stubborn searching.

The mentally retarded children most often took whatever was lying on top. They did not search for the required object and did not use the method of comparison by placing the blocks next to or on top of each other.

The normal seven year old, having given the experimenter the wrong object by mistake, was capable of realizing his error and tried to correct it. "I made a mistake," explained one of the children in a businesslike manner; "Here, take this one, now it's correct."

Mentally retarded children, as a rule, did not notice their mistakes and, consequently, made no attempt to correct them. The features big-small were handled by the mentally retarded with the least amount of errors. The concepts long, wide, high and thick led them to the concept of big, which is most often used in life and therefore was more familiar to the mentally retarded children. For this same reason the concepts high, low, narrow were more difficult for them, since in daily life the children experience them more rarely. They were replaced more than other concepts by the concepts big-small.

The ability to compare and establish which of the objects is bigger-smaller, longer-shorter, wider-narrower, thicker-thinner, higher-lower was evaluated using the same sets, namely, the sets of blocks, rulers, sticks and pencils.

In investigating orientation in the concepts bigger-smaller in size, two sets of blocks were used. The first set served to explain the task; the second was used to complete the task. The experimenter showed the pupil one of the blocks of the first set and asked the pupil to give him all the blocks in the second set that were larger or smaller than the block he held. The task was repeated three times using different blocks.

Using two identical sets of the counting ladder, knowledge of the concepts wider-narrower, thicker-thinner was evaluated.

Together with this and in the same order an investigation was conducted of orientation in the concepts of the same size, of the same length, width, etc. In these cases the child was required to give the experimenter only one object from a set, one equivalent to the given object. This task, like the preceding one, was repeated three times; each time the object of the assignment was changed. The children from the kindergarten performed all the above assignments completely.

Of the mentally retarded children, only one fully performed the assignment "give the same" (in size, width, height, etc.), 14 did it partially, and 15 did not do it. The assignment to give all objects smaller (or narrower, shorter, etc.) than the one shown was fully completed by only two pupils, partially completed by 14 and not done by 14. The assignment to give all objects larger (wider, longer, etc.) than the one shown was fully completed by three pupils, partially, by 14, and not performed by 13 pupils.

The most difficult was the assignment to select an object of the same size (width, height, etc.) from a second set. The difficulty was that the children could not isolate the required object, and thus selected objects which were close in size to the given object.

Research was further conducted in the orientation of children in the concepts _bigger-smaller_, _the same in size_, _longer-shorter_, _the same in length_, etc., when only one of the comparable objects was placed before the child, the other being compared mentally. This was done by placing the sets at different ends of the classroom. The pupil was given the assignment at one end of the room and performed it at the other. Under these conditions, the subject had to retain in his memory the image of the given object for a prolonged length of time. The mentally retarded children gave a significantly lower number of correct answers than in the preceding tasks.

Of the 30 auxiliary school pupils, only six handled the assignments partially, while the other 24 could not cope with a single assignment.

The children from the kindergarten gave 96% of the answers to this assignment correctly.

The general character of the performance of the assignments in the two examined groups of children varied. The normally developing children performed the assignment carefully, with great accuracy. They checked to see whether one more of the required objects did not still remain in the collection. Performance of the assignments was often accompanied by discourse: "I'll go look and see if one is still there."

The mentally retarded children were not interested in accuracy in their completion of the assignments. In the majority of cases, there was no discourse concerning the assignment.

The normally developing children, trying to make the completion of the task easier for themselves, introduced didactic material into the system. For example, of 15 children of the older group from the kindergarten, 12 first put the set into a definite system, then began to fulfill the assignment. In working with the criterion _length_, they arranged the sticks of the counting ladder into a little "ladder," making the ends of all the bars even on one side so that it was easier to see the difference in their lengths. To the question of why they did this, the children answered: "It is clearer this way," and "You can find it better this way."

As a rule, the mentally retarded children made no independent changes in the arrangement of the displayed material.

Having established that the counting bars could be arranged in a definite order (a "ladder"), the normal children, having received rulers in a scattered pile (in work with the criterion width), said: "Here, too, probably, you can make a little ladder," and immediately began to order the rulers.

The experimenter prompted the mentally retarded children by suggesting the possibility of arranging the sticks of the counting ladder in order (a "ladder"), thereby making the solution of the problem easier for themselves. Under the influence of such direct prompting, the children did this with the counting ladder, but were still unable to apply this method to complete a problem on other didactic material. During work with the rulers, the experimenter again had to remind the children of the possibility of putting the scattered material into order.

When comparing objects with each other, the normally developing children placed one object on another (for example, rulers) or placed one object beside another (for example, sticks of the counting ladder). The mentally retarded children did not do this. If the experimenter tried to direct them to an appropriate device, the children could zealously equate objects by a non-essential criterion. For example, the rulers that were of different widths but of equal lengths were zealously equated by length, when it was required to compare them by width.

Knowledge of Elementary Quantitative Relationships

We began the investigation of the children's knowledge of elementary quantitative relationships by ascertaining orientation in quantity of objects without counting them: many-few, more-less, as many as.

The investigation was conducted with the help of buttons of the same size, shape, and color. Here is the sequence of the assignments

1) The child was shown two groups of buttons and asked to indicate in which group there were many buttons, and in which there were few;

2) He was asked to take as many buttons as the experimenter took, then more, then less than the experimenter; he was also asked to take an equal amount in both hands;

3) He was required to determine in which of two sharply separated groups there were more buttons, and in which there were less;

4) He was asked to separate the buttons into two groups so that there would be more buttons in one of them, fewer buttons in the other, or so that there would be as many in one group as there were in the other.

Thus the child's orientation in the indicated concepts was checked by the actions of the child himself. The children from the kindergarten did all the assignments freely. The concept as many as was unfamiliar to only one of the 15 children who were examined.

Of the group of mentally retarded children, only three handled all the assignments. The others handled the assignments partially or not at all.

The concept as many as was the most difficult for them. This concept is rarely encountered in the everyday life of a child, in any case, not often enough for the mentally retarded child to be able to master it. Moreover, to master the correlations as many as and equally, the child must understand that to each unit of one quantity there must correspond only one unit of another quantity. This is complex for the mentally retarded child.

The concepts larger-smaller are also difficult. In mastering these concepts, the pupils of the auxiliary school can be seen to have a quantitative peculiarity. The concepts larger-smaller and many-few were not related to each other by these pupils. They know the words, but they do not connect the concepts designated by them in the pairs larger-smaller, large-small, long-short, and others, but perceive each of them as a separate quality of objects. This is confirmed by the number of correct answers obtained when investigating the indicated concepts (pairs of opposites). For example, in orientation in the concept many, 82% of the answers were correct and in the concept few, 68%; in the concept larger, 54% and in the concept smaller, 39% of the answers were correct.

To make these concepts more precise and to consolidate them in the consciousness of auxiliary school pupils in the first grade, it is necessary to return to them constantly during further instruction in arithmetic, and to broaden and deepen them.

As early as the study of numerical order the child should know practically (without verbal generalizations) that any number in numerical order is greater than the preceding one and smaller than the following one. Furthermore, all numbers in numerical order up to a given number are smaller than this number, and all numbers after it are larger.

The child should also know practically (by comparison, not naming other components of operations) that the sum is greater than either of the summands, that the minuend is greater than the subtrahend and greater than the remainder, and that, in counting by twos and by threes, there are more twos than threes in ten.

In the second grade these concepts acquire special significance when increasing and decreasing a number by several units, by so many more and by so many less. The concept must also be instilled in the child that in multiplying by an integer the obtained product is greater than the multiplicand, and the dividend is always greater than the divisor or the quotient. In studying measures of weight, length, and time it must be shown that one measure is greater than others.

In the third grade the concepts larger-smaller are encountered in the study of multiple comparisons.

Ability to Count to Ten

Investigation of the state of counting to ten was conducted in the following order:

1) We checked knowledge of the names of the numbers and the order in which they are found up to ten (counting without didactic material);

2) We checked the ability to count objects up to ten (the objects were buttons).

In the second case the children were asked to determine how many buttons there were in a box (ten buttons), and to pick out a specific number of buttons (numbers selected at random).

An analysis of the material obtained shows that the mentally retarded children lagged significantly behind the normally developing seven year old children.

The seven year old children from the older group of the kindergarten not only counted to ten, but significantly higher (in some cases to 100).

The mentally retarded children could name the numerals in order up to ten in 69% of the cases; they could count up to ten objects in 63% of the cases.

Ability to Perform Arithmetic Operations

Understanding of the operations of addition and subtraction was checked by the solving of very simple problems (story problems) and examples (number problems).

The pupils were shown four problems in one operation using the most simple verbal formation. Numbers in the problems did not exceed five. The texts of these problems follow.

1. "You had two pieces of candy, and your mama gave you another piece. How many pieces of candy did you have then?"

2. "Misha had three pictures. Vova gave him two more. How many pictures did Misha have?"

3. "A boy had two rabbits. One of them ran away. How many rabbits were left?"

4. "Shura had five pencils. He gave two pencils to a friend. How many pencils did he have left?"

The problems were given orally to the children, they repeated them, then solved them orally. The pupils were told in advance that, in case of difficulties in solving the problems, they could use didactic material.

In solving the examples, which were given after the problems, the possibility of using didactic material was again indicated.

The examples were analogous to the problems: 2 + 1; 3 + 2; 5 - 2; 2 - 1.

The mentally retarded children's solutions of the problems and, even more, of the abstract examples were sharply distinct from the solutions by the kindergarten children. In doing the operations in the problems and examples, the mentally retarded children gave only 50% of their answers correctly.

Without exception, all of the kindergarten children solved the problems and examples given them; they very seldom used didactic material (only one child, in one instance, "5 - 2", used his fingers).

One mentally retarded child who solved a problem counted on his fingers; another child counted on sticks.

In solving the abstract examples, all the mentally retarded children, without exception, made use of didactic material.

The problems in one operation had concrete conditions. It is understood that it is easier to solve a concrete problem with numbers from one to five than it is to solve an abstract example.

On the contrary, in the upper grades, when the conditions of a problem become complex, when logical work is required in order to understand the problem and its solution, then the solution of the abstract example becomes easier in comparison with the solution of the problem.

Summary

On the basis of the data obtained it may be concluded that:

1. The mentally retarded children from ages eight to twelve whom we investigated were not prepared, judging by the level of their knowledge, for instruction in arithmetic, despite the fact that many of them had already studied one or two years in the mass school. The intellectually normal children already had the corresponding preparation by the time they were seven years old.

2. Approaching instruction in arithmetic, one must first prepare the mentally retarded child to master the study material. Such preparation should be expressed in developing in the children an orientation in the spatial features of objects and in their quantitative relationships, as well as in the development of numerical conceptions.

3. Moreover, preparation for mastering the study material should be expressed in the development in children of the fundamental logical operations:

a) the ability to note similarities and differences of objects and their features;

b) the ability to abstract from separate features those aspects characteristic of the objects or phenomena;

c) the ability to generalize and form general conceptions and concepts;

d) the ability to classify and systematize;

e) the ability to plan.

The Formation of Conceptions of Spatial and Quantitative Relationships in Mentally Retarded Children

The preliminary investigation helped us note several ways of broadening mathematical conceptions and concepts in mentally retarded children when studying in a preparatory course, as well as several devices of study of numbers between 1 and 20. In this vein, we conducted an experimental investigation.

The investigation was conducted over a period of two years in the first grades of Leningrad's auxiliary school. Altogether, the investigation involved 101 subjects varying in age from eight to twelve years (eight years--six pupils, nine years--22 pupils, ten years--31 pupils, eleven years--33 pupils, and twelve years--nine pupils). Of these children, 70 had studied one or two years in the mass school or had been in kindergarten.

In order to establish the devices for visual instruction in arithmetic and to check their effectiveness, we conducted systematic observations of the work of the best teachers of the auxiliary schools of Leningrad, and scattered observations of the auxiliary schools of Moscow, as well as of the mass school No. 210 in Leningrad.

In this manner the work was done in ordinary school surroundings. In several cases individual lessons were necessarily conducted with the children.

As a result of our observations and experimental instruction, we established a series of techniques which facilitated both the conscious mastery of the study material in arithmetic, and the development of the thinking of the mentally retarded child.

In this work we have not posed the problem of giving a methodology of arithmetic instruction, but have tried only to establish various techniques for utilizing visual aids in the study of the fundamental sections of an elementary course in arithmetic.

Only by visual instruction did the child's orientation develop regarding spatial features such as *large-small*, *larger-smaller*, *of the same volume*, *long-short*, *longer-shorter*, *of the same length*, *wide-narrow*, *wider-narrower*, *of the same width*, *thick-thin*, *thicker-thinner*, *of the same thickness*, *high-low*, *higher-lower*, *of the same height*.

As our preliminary investigation showed, these concepts, are absent or insufficiently defined in many children who enter the first grade of the auxiliary school. In conducting the work we therefore concentrated upon developing in these children the above concepts.

Developing Concepts of Spatial Features

To form concepts that correspond to definite spatial features of objects and to compare objects according to such specific features, we used the same aids as those in our preliminary research:

1) blocks: in forming the concepts *large-small*, *larger-smaller*, *of the same size*;

2) the counting ladder, placed horizontally: in forming the concepts *long-short*, *longer-shorter*, *of the same length*; with the help of the same ladder, placed vertically, we developed the concepts *high-low*, *higher-lower*, *of the same height*;

3) rulers: in forming the concepts *wide-narrow*, *wider-narrower*, *of the same width*;

4) pencils: in forming the concepts *thick-thin*, *thicker-thinner*, *of the same thickness*.

The aids that we used were characterized by the fact that each set isolated a specific spatial feature. For example, the characteristic spatial feature of the block was its volume. One does not say of a block that it is long or short, wide or narrow. All three of its measurements are mutually equal. The most characteristic distinguishing feature of the block is its size: *large* or *small*.

At first we taught the children to distinguish natural objects in surrounding life according to their spatial features. This work was conducted during a walk or excursion with the children in a garden, a park or on a street. The pupils found a large house and a small house, a high and a low tree. In natural objects, however, features are given in combinations: a tree is not only high, but also thick; a house is high but narrow in comparison with its neighbor--a low, but very long house. For the mentally retarded child, this extraordinarily complicates the comparison of objects according to any one feature. To isolate one or another feature from their entire complexity is often beyond the powers of the auxiliary school pupil at the time of his first instruction.

The mentally retarded children entering the first grade of the auxiliary school could not perform the most elementary comparison of objects. For example, the pupils could compare the thickness of two pencils only when the lengths of the pencils were identical. When they were not identical, the pupils confused a thin pencil with a short one. Obviously they did not have any clear concept of width and length, since they confused these features.

To help the mentally retarded children to make the simplest comparison of objects and to teach them to isolate the feature necessary for comparison from all other features, aids were required in which all an object's features were equal except one. For example, the feature long-short is isolated with the help of a counting ladder, in which the bars are equal in all features except length. The feature of width is isolated with the help of rulers in length and thickness but different in width, etc.

Thus isolation, with the help of an appropriate aid, from a combination of features facilitates the apperception of this feature by the mentally retarded child and, by the same token, facilitates the formation of the corresponding concept.

This is the first distinctive characteristic of the methods of work which we have proposed.

The second distinguishing feature of these methods is the possibility of gradually widening the scope of the child's attention, because of the large number of objects in each set and the system of its construction. As a rule, we began working with the mentally retarded children in the comparison of two objects and, adding one more object to each object in the comparison, gradually increased their

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number, eventually offering the whole set. When there was a relapse, with increased mistakes (as the child, working with a whole set, found it again difficult in some cases to grasp all the objects of the set), we again reduced the number of objects.

In the children's work on the counting ladder, we noticed that they compared bars of 1, 2 and 3 centimeters easily, and that it was hard for them to compare bars of 8, 9 and 10 centimeters. The same was observed during work with the rulers, blocks and other objects. This can be explained by the fact that the large objects of a set were less distinct from each other than the small objects. The relative difference in the sizes of the former is less than in the sizes of the latter. For example, in the counting ladder a 2 decimeter bar is twice as long as a 1 decimeter bar, but a 10 decimeter bar is only $1\frac{1}{9}$ as long as a 9 decimeter bar. Consequently, in the first case the difference is more noticeable and is thus more easily established.

To facilitate the gradual transition to the comparison of objects which are hard for the children to distinguish, we used the following device. First we proposed bars of 1 decimeter and 10 decimetersfor comparison, then 1 decimeter and 9 decimeter bars; 3 decimeter and 9 decimeter; 2 decimeter and 8 decimeter; 3 decimeter and 8 decimeter, etc. Gradually reducing the difference in the lengths of the bars being compared, we led the children to the comparison of two, and then of several, bars which which were slightly different in length.

In including the entire set as a whole into the work, we began the comparison with small objects, gradually passing on to large ones.

The transition from objects with a greater disparity in size to those with less disparity, that is, the gradual increase in the difficulty in distinguishing the compared objects, is the third distinctive feature of the methods of work which we proposed.

All these techniques, as well as several special techniques, shall be described in more detail in the account of the separate stages of the investigation, to which we shall now turn.

We began our instruction of the orientation in spatial features with the features small-large, the most easily understood by the children.

Formation of the corresponding concepts was first conducted on one set of ten blocks with edges of 1 centimeter to 10 centimeters.

For convenience in the future development of this statement, let us number the blocks: of 1 centimeter--#1, of 2 centimeters--#2, 3 centimeters--#3, etc. Pupils were given the blocks in no particular order (at random).

First we conducted the comparison of two blocks of 1 centimeter and 10 centimeters; then of 1 centimeter and 9 centimeters; 2 centimeters and 9 centimeters and so on. In the approach to comparing blocks #5 and #6, already difficult to distinguish by size, the children were offered blocks #1, #2, #3; that is, the number of blocks was increased. Then blocks #1, #2, #3, #4, etc. were taken.

After we had conducted the comparison of two, three and more blocks of the greatest contrasting size, the whole set was presented. The blocks were placed so that they were visible to all the children. A slightly inclined pedestal was used for this, which was placed on the table.

One pupil was called up to the table. The teacher gave him a problem: "Give me the smallest block." After the pupil had given him one of the blocks from the table, the teacher gave him the following problem: "Give me the smallest of the remaining blocks." This was continued until the whole set had been used up. All pupils were called in turn.

Then the assignment was changed: "Give me the largest block," "Give me the largest of the remaining blocks." And again the whole set was used.

In those cases in which, despite the preliminary work in comparing two, three, and more blocks, gradually increasing their number, completion of the assignment on the whole set again troubled the pupil, a special device was used. Because the difficulty was connected with the large number of blocks, from which one had to be selected, the teacher limited the number of blocks to three or four. By the same token the child's attention was concentrated on narrower material. This method decreased the number of blocks, unlike the method which began comparison with a small number of objects and then gradually enlarged this number.

To show how the work proceeded in these cases, we produce here excerpts from appropriate reports of the lessons.

> Ten blocks are in front of the child in random order. He is asked to select the smallest block, then the smallest of the remaining ones. Both assignments are done correctly. But in doing the following analogous assignment, the pupil errs: he selects block #6 instead of #3. Then the teacher removes certain blocks, leaving only blocks #3, #4, #5, and #6 before the pupil; the assignment is repeated: select the smallest of them. The pupil gives block #3.

Thus the technique of reducing the number of objects gives a positive result.

> The other assignments involving blocks #4, #5, and #6 were done without errors. Then the remaining blocks #7, #8, #9, and #10 were included in the assignments until the whole set was used up.
>
> After this, assignments using the whole set were again attempted, in which the pupil, comparing blocks #8, #9, and #10, again made a mistake. In the subsequent completion of the assignment using the whole set, however, there were no mistakes.

In different pupils other mistakes were encountered; when selecting the smallest block (#1) from the set, the child gave the teacher the first small block that caught his eye. The expression "the smallest" was understandable to the children, since the ending -est had been explained separately, before the work with the blocks (the sweetest, sourest, prettiest, cleanest, etc.).

If we turn to the materials of the investigation, we see that normally developing children, before beginning to do the assignment (for example, "Give me the smallest block"), generally on their own initiative, put the set in order before doing the assignment, trying to make the selection of the required object easier for themselves. They seemed to be planning their work.

The mentally retarded children must be led to the correct solution of the problem along the same road traveled by the normally developing children. Hence to correct the mistakes mentioned above, the pupils were asked to put all the blocks in order "by height." Having done this, they easily found the smallest block in the set, then the smallest of the remaining blocks. Using such a technique, the teacher obtained positive results.

There were cases in which the pupil could not place the blocks "by height." Then the teacher himself positioned the blocks, showing the order of their arrangement. Having done this several times, the teacher asked the pupil to do the same thing and aided him in case of difficulty.

Having arranged the blocks "by height," the pupil found the smallest block without difficulty. The system in which the child arranged the blocks made it possible for him to perform correctly all the teacher's other assignments.

There were cases in which the pupil made a mistake, not knowing what the word small meant, and instead of giving the teacher the smallest block, he gave him the largest block in the set, or one of the largest. In these cases the teacher indicated the smallest block to the pupils and named it, then indicated and named the smallest of the remaining blocks, and so on. The set was presented to the pupils in a specific order: all blocks were placed "by height." After the teacher had gradually gone through the whole set in his demonstration, the pupils were asked to repeat what the teacher had done.

After one or two demonstrations the pupils usually mastered the concept small. The same work was conducted with the concept large.

During the work described above, leading questions were asked: "Isn't there another block smaller than this one?" "Take a good look, there is a block smaller than this one there." "Did you look at all the blocks?"

Such questions and advice helped the pupils. The teacher's words organized the pupils' actions and directed their attention to the necessary object.

The normally developing children usually sought the required object actively, by themselves. They were not contented until they had checked all the blocks shown to them. "Isn't there a smaller one?"--the children explained their searchings.

As a rule the mentally retarded children did not search for the object corresponding to the assignment. They were not interested in the possible existence in the set of an object more closely corresponding to the assignment. They did not even guess the possibility of a more accurate way to do the assignment. The mentally retarded children usually selected whatever fell into their hands, whatever was closest to them.

To the techniques for helping develop in pupils an active relationship and interest to the assignment they are doing, one may also add the one in which the required block is placed by the teacher under the other blocks so that it is not visible to the child. Then the teacher's questions and advice facilitate the pupil's search. In this way they find the appropriate object, which was not immediately seen and which had to be searched for.

The techniques described were also used by us in working with pupils using the counting ladder and the set of rulers and pencils (forming the concepts long-short, wide-narrow, thick-thin, high-low).

In several of the auxiliary school pupils we observed a lack of self-confidence when they were doing the assignments. The teacher constantly had to confirm the correctness of their performance of the assignments.

Having learned to isolate one or another spatial feature of objects in the appropriate sets, the pupils then dealt with them in work with ordinary objects.

After the pupils had learned to isolate objects according to their size (large-small, long-short, etc.), they were given a new assignment, connected with the formation of the concepts more-less, longer-shorter, wider-narrower, thicker-thinner, higher-lower and the concept of the same length, width, etc.

The ability to compare objects developed during formation of these concepts.

Fulfillment of the preceding assignments in each separate case was connected with a search for one object. In doing the new assignment it was necessary to search not for one object, but for several. Thus this task is somewhat more complex as compared with the preceding ones.

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Work was conducted using the same sets as before. The counting ladder was also among the sets. The teacher put its bars in order. For convenience in the further description of this assignment, we shall number them thus: 1 decimeter--#1, 2 decimeter--#2, 3 decimeters #3, etc.

A pupil was called up to the table. The teacher took one of the bars, say #4, and told the pupil: "Give me all the sticks shorter than this one." When one bar was taken out, the set was divided into two parts: in one of them all the bars were shorter than the one the teacher had, and, in the other, all were longer than this bar.

The mentally retarded children as a rule did not at first give all the bars required by the assignment but limited themselves to selecting one or two of them which stood out most sharply from the rest. They did not complete the assignment.

The normally developing children did the assignment more carefully. They gave the teacher all the required bars and always tried to check whether there were not still others which would correspond to the assignment.

The mentally retarded children were shown the method of searching for the required objects by placing the given model on top of or next to all objects of the set in turn. This method allowed one to check the accuracy of the assignment's performance.

After a great many exercises there was developed in the pupils the ability to compare objects by length independently (without help from the teacher) by placing one on top of another, and to check the correctness of their conclusion.

After the pupils had mastered the techniques for comparing objects when working with one set, they were given a more complex assignment in which another set, analogous to the first, was included.

Work was conducted on the two sets in this manner: the teacher gave the assignment on one, the pupil carried it out on the other. The pupil's set lay on the table with its elements in no particular order. The teacher's set was placed on the same table and was shown to the pupil in appropriate order.

The work's complication consisted in the lack of an orientator in the pupil's set. Giving the assignment, the teacher took a block from his set, thus dividing the remaining blocks into two parts: in one all the blocks were smaller than the given one, and in the other they were larger. The pupil, using his own set, had to establish the blocks which corresponded to the assignment by comparing the model and the entire set of blocks. This was difficult for him, and he often made mistakes.

Moreover, the number of objects for comparison was increased, since another new object was added--the block or stick from the teacher's set. This also hindered the performance of the assignment, since for the mentally retarded child theappearance of a new object in any structure creates new difficulties in mastering this structure.

The pupils were reminded that in searching for the required block or stick they could compare them by placing one on top of or next to another. But here they encountered another difficulty. The comparison of two objects, even by placing one on top of the other, was beyond the auxiliary school pupils',capacity in some cases, since they compared objects by features which were not essential to the given assignment; in some cases they did not know how to look for the essential feature even when it was isolated. For example, in the assignment to show which of two rulers was narrower, the children, comparing the rulers, painstakingly evened all the ruler's ends, paying no attention to the difference in width.

To correct similar errors, the teacher asked the pupils not only to look at, but to feel with their hands the difference in width that could noted when one ruler was placed on top of another. Analogous work was conducted on the other sets (the counting ladder, the set of blocks, etc.)

The next variation which complicated the fulfillment of the assignment was a change in the set's color. Instead of a counting ladder whose sticks were divided into various colored sections, a set painted all the same color was used. It was more difficult to do the assignment using the set of one color, since the sticks were no longer divided into parts according to length.

In the work with a uniformly colored set, the number of errors increased. The children did not distinguish sticks which were a little longer or shorter than the given stick from sticks which were equal to the given one.

The new coloration of the set disoriented the pupils. They began comparing the sticks of such a set as if they were doing a new, unfamiliar assignment. It was again necessary to conduct a series of exercises with them before they learned to do the given assignment. A new technique also had to be introduced, to establish a definite system for comparing the given model with the sticks from the set on which the assignment was done. This technique was to advise the pupils to compare the sticks, beginning with the shortest stick (#1) and to place the model in turn next to all the sticks in the set (to stick #2, #3, #4, etc.), until the unknown was found. As a result of several exercises, the pupils learned to do this assignment without errors.

It must be remarked that, in doing analogous work with normally developing children, the change in the set's coloration had no significant value.

Finally another complication was introduced in the assignment. Two sets on which sizes were compared were placed at a distance from each other. One set was placed on the table near the teacher; the second was carried to the other end of the classroom. Both sets were of the same color. The assigned object was shown to the child, and he was allowed to hold it in his hands; then it was taken by the teacher. The child had to walk across the classroom during this time, retaining in his memory the image of the object shown to him.

The new assignment produced a sharp increase in the number of errors. Only a very few pupils could do the assignment completely. Several pupils did the assignment only partially.

To make the assignment easier for the children, one set was placed significantly close to the other, and was gradually moved away as the pupils became familiar with doing the assignment at the short distance. Moreover, when doing the assignments the pupils were aided by repeated demonstration of the object.

Only as a result of many exercises, however, did the children learn to perform this assignment correctly.

Developing Concepts of Quantitative Relationships

The fundamental quantitative relationships (more-fewer, as many as, and equally) are mastered by the children during their study of the first ten numbers. In the first grade of the auxiliary school the essence of the given concepts is explained. More and fewer are studied as mutually contradictory concepts. As many as and equally are studied as equivalent concepts.

The concepts more and fewer, to a certain degree, have already been introduced to the children. Comparison of objects by size, and comparison by quantity, however, involves a greater distinction, which we should introduce to the child's consciousness. This distinction consists of the following: in the first case one object is compared with another; in the second case a set of objects is compared.

We first investigated the techniques connected with the formation of the concepts more-fewer in comparing sets of natural objects. Used as visual aids were objects of the child's study, life and play, as well as sticks, paper circles and pictures illustrating groups of objects.

The work was begun by showing the children two separate groups of objects and asking them to determine which group had more and which had fewer objects. At first the majority of the children had difficulty in answering the questions posed.

We have already noted cases in which a mentally retarded child had difficulty in comparing two objects if, in addition to the feature by which he was comparing them, the objects differed in some other feature; for example, difference in color hindered comparison of objects by size.

Comparison of groups of objects by quantity was therefore done at first on homogeneous objects, the other conditions remained the same, only their quantity changed and was to be determined.

To show how the work was conducted, we give here an extract of the report from one lesson:

On the teacher's table there is a glass containing 6 pencils and a box with 3 pencils. All the pencils are yellow. The teacher asks: "Children, look carefully at the pencils and tell me where there are more pencils." Several pupils give correct answers, some answer confusedly, unsurely. Many pupils give incorrect answers. The teacher calls the pupils who gave correct answers one at a time up to the table. They then show their classmates the container in which there are more pencils and that in which there are fewer.

The same work was done using little flags. Five red flags were put into the glass and 2 red flags into the box.

The work proceeded analogously.

Even though the exercises were repeated several times, some of the pupils still did not master the concepts more and fewer.

The teacher gave groups which contrasted more sharply. In the glass he put 10 pencils and in the box, 2 of the same kind of pencils.

The majority of the pupils gave the correct answers.

Similar illustrations were conducted with other objects whose quantity was sharply different; for example, 15 blue flags were compared with 3 blue flags.

To show the work process better, below we give an excerpt from a lesson report concerning the formation of the concept more in quantity:

The teacher checks the children's knowledge of the basic colors.

Teacher: What color is this pencil? (Shows the children a red pencil.)

The pupils answer correctly. The teacher takes 10 red pencils in one hand and 3 blue ones in the other.

Teacher: Which pencils are there more of?

Pupil: There are more red pencils.

T: Come, show me where there are more.

The pupil comes up to the teacher and points to the group of red pencils.

T: Here (points to the red pencils), of these pencils there are ...

P: More.

T: And here? She points to the blue ones.

P: Here there are fewer pencils.

But not all the pupils give the correct answers. Some confuse the concepts more and fewer. A pupil is called up to the table.

T: Where are there more pencils?

P: Here there are more. (He points to the 3 blue pencils)

T: And here?

P: Here there are fewer. He points to the red pencils, then quickly corrects his answer. No, there are more here.

T: Where are there more here?

P: There are more here, than here. He points again to the red pencils.

T: That is, where are there more pencils? Show me. The pupil points to the 3 blue pencils.

It is evident from the pupil's answers that it makes no difference to him which group the concepts more or fewer are related to; that is, he does not distinguish these concepts. Much work is required to teach him to distinguish them.

The pupil is given several consecutive tasks:

The teacher takes 9 blue flags and 2 red flags and holds them so that the whole class sees the number of both types of flag.

T: Which flags do I have more of?

P: You have more blue flags.

Several pupils repeat this answer, coming up to the table and pointing to the blue flags. Then the teacher calls up another pupil and gives him an assignment: Give these pupils the red flags, and these, the blue ones. He points to rows of desks. The pupil passes out the flags.

T: Everyone who got a red flag, raise your hand.

The children sitting in the first two rows raise their hands; they all have red flags.

T: Everyone who has a blue flag, raise your hand.

The children sitting at the desks of the third row raise their hands.

T: Which flags are there more of?

P: There are more red flags.

Similar exercises are conducted with textbooks, notebooks, and toys.

Then the teacher fastens a picture depicting a boy and a girl to the blackboard. The boy is holding a bunch of 10 carrots and the girl, three carrots.

T: Who has more carrots?

P: The boy has more carrots.

The teacher fastens another picture to the blackboard. The picture depicts a river with children bathing in it. There are many children in the water, splashing and swimming. Two girls are sitting on the bank.

T: Where are there more children, in the water or on the shore?

P: More are swimming. There are more children in the water.

T: More than where?

P: There are more children in the water than there are on the shore.

The answer is given with the teacher's help.

The assignment was made more complex. The teacher showed the children a mixed group of red and blue pencils in which there were significantly more red pencils than blue ones. The pupils were asked to determine which type of pencil there were more of. They had to separate from the mixed group the greater number of objects. This was difficult for the children and they made many mistakes, which could be corrected only by separating the mixed group of pencils into separate groups according to color. This work was also conducted with the flags.

Then a picture was shown which depicted children in a forest gathering mushrooms. Some were clearly visible. It could be seen that two of them were boys and that the other seven were girls (they wore dresses).

T: Who are there more of in the forest, boys or girls?

P: There are more girls.

After the children had learned to separate the obviously larger group, the difference in number of objects was gradually decreased. For example, two groups were taken, eight objects in one group and three in the other, then 8 and 4, 7 and 4, 7 and 5, 6 and 5. In such a gradual transition the children learned to compare and distinguish groups of objects which were not significantly different in quantity.

When the children had learned to compare two groups, the teacher asked them to compare three, four, and several groups. The children determined in which there were more objects and in which there were fewer.

There the children had to compare quantities by counting them. This is a difficult assignment, since it requires the ability to count and a knowledge of numbers.

As a visual aid in the work with the whole class the following objects were used: red and blue pencils (ten of each), red and blue flags (ten of each), a collection of white and red paper circles of different diameters (ten of each color), and pictures. Red circles and circles of various colors and sizes placed in envelopes or boxes served as distribution material for the pupils' work.

The work was first conducted on identical objects, in which all features were equal except for quantity (pencils or paper circles of one color). After the children had mastered the comparison of homogeneous groups, they were given objects with different colorings. At first groups which were sharply distinct from each other were presented, later the groups were more similar.

Gradually the children began to be able to compare groups containing a close number of elements (2 and 7; 3 and 7; 2 and 6; 3 and 6; 4 and 6; 3 and 5; 4 and 5; 6 and 7).

The most important techniques were those facilitating instruction of the mentally retarded child in abstracting himself from all the features of the objects except their number. The ability to abstract is weakly developed in the mentally retarded children. In the experiment we found that it was difficult for them to abstract themselves from the size of objects when they had to compare groups by size.

To develop this ability in children, appropriate techniques were used; they formed a gradual transition from the comparison of two groups of homogeneous objects to the comparison of two groups of nonhomogeneous objects, first distinguished by color, then by size, etc.

The objects of comparison were buttons, sticks, leaves collected by the children in the park, squares and circles of varicolored paper.

In mastering arithmetic concepts, the children were troubled by the words as many as. They understood, for example, that the number of circles in the groups was identical, but the phrase itself was incomprehensible to them. Hence special attention must be directed to the mastery of this phrase.

When the pupils had mastered the concept as many as to some extent, the teacher presented a somewhat more complicated new problem in which circles differing in size and color were used. For a long time the children were still inclined to compare the groups not by the quantity of objects but by their size. Therefore, when the circles were changed in size there appeared many erroneous answers. To avoid them, work was conducted with the children analogous to that mentioned above in describing the process of forming the concepts more-less in quantity.

When the children had learned to compare two groups of objects by size, they were asked to compare three groups with each other, then four groups, then more groups. The children gradually acquired the ability to name all the numbers larger or smaller than any number up to ten; for example, all numbers up to ten were larger than five, etc.

With the help of a special technique, the children were taught to compare sets, one of which was real and one of which was imagined. We give here the materials from one lesson to show how this device was used:

> Each pupil had a set of buttons consisting of 10 large black buttons and 10 small white buttons. The teacher had bunches of 2, 3, 4, and 5 buttons tied together with thread. He also had sets of large black buttons, small white buttons and a mixed set.
>
> The teacher showed the children three large black buttons.
>
> T: Take the white buttons. Put more of them on your desk than I have. Cover them with your notebook.
>
> The children do the assignment correctly, each putting down 4 or 5 buttons.
>
> T: How many buttons do I have?
>
> P: You have three buttons.
>
> T: And how many buttons do you have?
>
> P: I have five buttons.
>
> T: Who has more buttons?
>
> The children give the correct answers. The teacher goes on to a new assignment.
>
> T: Now I will show you some buttons and you tell me how many buttons you are putting on the desk (shows 2 large buttons). You must put more down. How many did you put down?

P: I put down five buttons.

The teacher repeats similar assignments several times with different numbers. The children answer correctly.

T: I have in my hand (shows his fist, in which he holds the buttons) five buttons; you put down fewer.

The pupils do the assignment.

T: How many did you put down?

P: I put down three buttons.

T: And if I had four and you had as many, how many would you have?

P: Also four.

T: We both have an equal amount. I have three and you have how many?

P: Also three.

To drill the studied material various games were used, for example, the game "silence" (with numbers). The teacher told the pupils earlier to lay out the number cards with larger, or, on the other hand, smaller numbers than the teacher had. The teacher either showed a number card or wrote a number on the blackboard. The pupils laid out their cards like a number cashbox.

Sketches and notations in the notebooks were also used.

We list here some sample assignments done in this period of work.

1. Draw some small circles. Draw more on the right and fewer on the left.
2. Draw as many circles on the right as there are on the left.
3. Write the numbers with figures. Write a smaller number on the left and a larger one on the right.
4. Write the numbers with figures. Put on the right as many as there are on the left.

The concepts *more*, *less*, *equal*, and *as many as* were also drilled during lessons in manual training, in which the children made various objects (fruits, vegetables, etc.) from clay. The children were given the assignments: "Make two pears, and more apples than pears"; "Make an equal number of beets and carrots"; "Cut three squares out of red paper and fewer out of white paper. Glue an equal number of each in your notebook." Here the selection of the quantity of represented objects was left up to the pupil himself.

Independent work developed the children's independent activity and creativity. It coordinated arithmetic knowledge with the child's labor and thus nicely consolidated the two.

Each number and arithmetic operation within the limits of ten was studied separately. Each such study consisted of the following sequence: formation of the given number; addition by adding on, and, somewhat later, subtraction by taking off; and composition and learning of tables of addition and subtraction.

The work practice of several of the teachers showed that this method was completely within the powers of the auxiliary school pupils and led to positive results.

The first number with which the pupils were acquainted was the unit. It was isolated by comparing a set of homogeneous objects and one of them.

In the auxiliary school, a necessary condition for isolating the unit from the set was the homogeneity of those objects with whose help this process was illustrated.

Mastering the rule of forming numbers is connected with the ability to generalize isolated concrete cases of forming numbers of the natural order. The auxiliary school pupils made these generalizations with difficulty; therefore, they came up against significant difficulties in passing from the visual formation of separate numbers to the concept that each number in a series is formed by adding a unit to the preceding number.

To help the auxiliary school pupils master the rule for forming numbers up to ten, precisely the same operation must be repeated on different objects, thereby enabling the accumulation of a large number of facts for their subsequent generalization.

The first condition for studying the rule of forming numbers and, consequently, the first method of work in this direction, is a demonstration of the formation of each new quantity, using homogeneous objects.

The mentally retarded child makes the same simple generalizations with difficulty. Each instance of forming a number, which is done on objects, is unrelated by the pupil to other similar instances; he perceives each instance in complete isolation. The mentally retarded child is not in a position to note that each time in forming, for example, a group of *six* objects, *five* objects are taken and another *one* similar object is added to them. To help the pupil notice this regularity, the whole process of forming numbers is divided into separate stages.

For demonstrating to the child that the number *six* can be formed not only by identical objects, but also by objects of different colors, forms, and sizes, the teacher uses objects having a difference in color and in other features.

Developing the Ability to Count to Ten

The child must count to determine any quantity. Often the auxiliary school teachers must observe the unique counting of the mentally retarded children; the pupils cannot count a group of eight or nine objects if they are positioned not in a row, but at random, without the possibility of rearrangement. When these same objects are positioned in a row, the pupils count them quite freely.

We present here an example of such counting. The teacher gives the assignment: "Count how many apples are in the dish." The pupil, very carefully touching each apple with his finger, counts, but omits some of the apples in the count and counts some of the apples twice. An incorrect result is obtained.

When counting, the child observed no system. He did not count in order, but picked out apples now from one place, now from another. Beginning to re-count several times, he began on a row of apples from various sides; he made mistakes, and obtained an incorrect result. Finally, after repeated attempts to count the apples, he accidentally counted them correctly and settled down.

The main cause for the children's difficulty in counting is the structural complexity of the group of objects presented for counting. If the material is not set in any system, the children themselves cannot determine it. The given case is analogous to the case in which the selection of a small or large block was facilitated by the presence of a system in the set of objects. The same applies in this case, in which it was necessary to systematize the material. In the opposite case, the children did not complete the assignment given them. This was shown by their questions; the children asked, "How do we count?" and "Where should we begin, from here or from here?"

Counting of the objects was facilitated by dividing the group into smaller, structurally simpler elements. For example, on the picture showing apples in a dish there were glued strips of paper so that all the apples were divided into sections, each of which was easily visible. The strips divided a complex quantitative group into simple groups. The pupils first counted the apples in one simple group, then, continuing to count, passed on to the second and third groups. They handled the counting of the entire "large" group in this way.

To teach the children to establish a system in counting objects in a complex group, we also used a table on which rows of circles were gradually rearranged. One after another group of circles passed before the pupils, beginning with a simple group having a system, and ending with a complex group, which had no system. Passing from one group to another, the pupil retained the order of counting that he had accepted in counting the circles of the first group, that is, from left to right. Dislocation of the circles in altering the direct order showed the pupil the direction of motion from top to bottom and vice-versa. As a result of such work, the children mastered the device of counting "complex" groupings.

Conducting the experimental work, we encountered still another characteristic of mentally retarded children which created difficulties in teaching them to count. For a long time they could not determine the number of objects if the numbers were given to them at random, for example, "give me 5 buttons," "give me 7 buttons," etc. The pupils tried to give the teacher all the buttons placed in front of them; they gathered up as many as their hands could hold.

In the investigation of the normally developing children such mistakes were not found.

To correct this flaw in counting, a technique was used in which the teacher's oral assignment was accompanied by numbers being laid out. The child himself found the number. As a result of many repetitions of such exercises, the children were taught to do correctly the oral assignment requiring them to give the teacher a specified amount of objects.

The indicated techniques were used during all the work in developing numerical concepts and arithmetic operations in the children.

The designation of a number by a numeral played a large role in forming numerical conceptions. But the auxiliary school pupils did not first link the number with the numeral or the numeral with the number. The pupils had to be given practice in designating quantities by numerals for example, the pupils are given one of the numerals and a box with a set of cards depicting groups of various numbers of homogeneous objects. The pupils must sort out all cards with a number of objects corresponding to the assigned numeral.

Another variant of exercises of this type is the following work. The pupils are given a set of cards on which are depicted various groups of homogeneous objects, and a set of number-cards. The child must sort out the numeral that corresponds to each group of objects.

A third variant could be to sort out a number of separate objects corresponding to the given numeral. The teacher asks the children to take one of the numerals and to select groups of various objects whose number would correspond to the given numeral.

All the indicated techniques help the child connect the numeral with the corresponding number of objects.

Developing Ability to Perform Arithmetic Operations

When work on one or another number is completed and the children have mastered counting within the limits of the given number, one must pass on to the study of operations within its limits.

As early as the formation of conceptions of the numbers two and three, the children encounter the operations of addition and subtraction. At this time the teacher reveals the idea of each operation.

The teacher should select visual aids for studying the arithmetic operations with special carefulness. Mastery of the calculating devices is facilitated only by techniques which aid in illustrating the addition of each separate unit of a second summand and allow the units to be formed into any groups. In other words, in studying arithmetic operations, moveable counting material must be used.

In studying addition and subtraction, the primary visual aids are the equipment of studying, as well as objects of the child's life and play -- for example, pencils, rulers, flags, checkers, spools, linen buttons, spoons.

Large circles for the blackboard (10 centimeters diameter) and small ones for the pupil (3 centimeters diameter) can serve as an aid. In addition, it is necessary to have sets of large and small digits for the teacher and pupil.

Besides the requirements stated above, which must be fulfilled by aids in studying the arithmetic operations, it is necessary that the first and second term be indicated boldly in each of them; for example, yellow and black pencils, red and blue flags, teaspoons and tablespoons, different-sized buttons, spools of different colors. Colored paper circles are a very useful aid.

An aid which clearly reveals the sense of arithmetic operations is the constructive picture (working model).

Study of the operation of addition and subtraction is begun by disclosing and explaining the meaning of each. The sense of the arithmetic operation is understood by the mentally retarded child better if the operation is connected with a specific concrete situation, especially if this situation is connected with the child himself, if he is included in it.

Moreover, the very process of joining, adding objects or, on the other hand, of taking away part of a general quantity must be shown very clearly to the mentally retarded child, with no digressive motions, so that the child will see only what he must master.

After repeated exercises in solving addition problem in which the terms are illustrated by objects, it is usual to begin solving examples with abstract numbers. The teacher writes them on the blackboard with the help of figure cut-outs and reads what he has written; for example, "One added to two equals what?" The pupils solve the example, writing the sign for "equals" and the answer.

Then at the teacher's dictation (without writing on the blackboard) the children compose examples from their own number cut-outs and solve them at their desks.

Having learned the sense of addition well, the children handle simple problems and examples easily. At first they add by ones. Then they add by groups of two and three and count by twos and by threes.

Here the teacher encounters great difficulties. The mentally retarded children have a stubborn tendency to perform addition by a factual unification of the objects into one common group, and then an enumeration of this group.

It is certainly easier to do the operation by counting, but this means that the children do not learn to calculate. To help the mentally retarded child master the calculating technique of adding a second term one unit at a time, the counting prop is used. In adding, the first term is given in the form of beads placed on a wire. The second term is represented by separate beads placed in a row. Having determined how many beads are on the wire and how many must be added to the given number, the child begins addition. But it is impossible for the pupil to place all beads onto the wire at once; he must put them on one at a time. By several repetitions of this process in adding various numbers the child masters the device of adding a second term by ones.

After the pupils have mastered this calculating technique and use it in adding various objects, the counting material is taken from their desks. The teacher asks them to solve an example without the aid of the counting material. But it is difficult to pass on to this immediately and, in order to facilitate this process, a table of numerical order is used.[1]

This aid consists of cut-out numbers, fastened close together in numerical order to the upper frame of the class blackboard. The classroom chart depicts the numerical order, and each pupil has a chart prepared by the teacher on an ordinary sheet of paper. The pupils must add two to five. They place their finger on the number "5," then move it to "6," saying, "one added to five is six," then move their finger to "7" and say, "one added to six is seven"; in sum: "Two added to five is seven." Such a performance of the operation is based on the use of sequential addition. In the future the children skim the numerical order on the blackboard with their eyes only, and no longer handle any kind of aid. A gradual abstraction from concrete objects for counting occurs. At this stage many pupils begin to add numbers mentally. They even close their eyes when calculating. In the more difficult cases (for example, "5 + 4"), some children return to sticks and the hand tables, but this occurs more and more rarely. Most of the children begin to count without visual material.

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[1] The aid mentioned and the methodology of working with it were borrowed by us from T. S. Yakubovskaya (teacher in Auxiliary School No. 1 of Leningrad).

The tables described have two other positive qualities: first, they aid the consolidation of numerical order in the pupil's memory; second, work with them facilitates performance of the assignment. In performing operations on sticks, the child must retain two series of numbers in his mind: the obtained intermediate result from adding the first unit, and the remaining units of the second term. Using the tables the child must remember only one number--the remaining units of the second term--since the table shows him the intermediate result from adding the first unit. Italso shows him the final result.

We consider this aid very valuable. All the children of the class where it was used mastered abstract calculation within the limits of ten rapidly and firmly.

After the addition of "2," the teacher instructs the children in adding "3," "4," etc., one at a time, grouping the units of the second term. For example, the assignment "4 + 3" can be done thus: "4 + 2 + 1" or "4 + 1 + 2."

In passing on to the addition of two terms of which the second is larger than the first, one must proceed using the technique of inverting the terms.** But this technique may be used only when the children know how to solve both examples (for example, "4 + 2 = 6" and "2 + 4 = 6"), and are convinced on the basis of calculation that the sum in both cases is the same and that it is easier to add on a smaller number than a larger one.

Although numerical figures are unsuitable to use in mastering the calculating techniques, since they hinder instruction, they are a useful aid in studying the composition of the number. In our opinion Lay's figures are the most precise numerical figures. The

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**The commutative principle (Ed.).

principle of constructing these figures is very simple. In studying the composition of the number, circles painted two contrasting colors (for example, black and green or black and red) are used. A precise expression of the composition of the number can be obtained.

A good method for studying the composition of numbers are sets of charts with buttons. Each pupil has two charts and a set of ten buttons of various sizes and identical color. The charts should be of such a width that two buttons could not be placed on them in a row, that is, just wide enough for one button. Blackboard charts should be placed side by side so as to make it possible gradually to incorporate into the system all cases of development of combination in the same order that the child sees at his desk.

The pupil is first asked to put all the buttons onto one chart, then onto both, then again onto both, but in a different combination, etc.

To show how this work was conducted, we give here a notation of the report from one lesson:

> The pupils have sets of buttons and charts. The teacher has circles of 5 decimeter diameter. The teacher gives the assignment: "Take one chart. Put six buttons on it. Put the other buttons in the box. Close the box and put it aside." The pupils do this. The teacher walks around the class, checks, then affixes colored circles to the blackboard (6 circles).
>
> T: How many buttons did you put on the chart?
>
> P: I put down six buttons.
>
> T: Take another chart and place six buttons on the two charts. (The teacher walks around the class and calls on a pupil who has put five buttons on one chart and one on the other.)
>
> T: How are your buttons set out? How many do you have on one chart and how many on the other?
>
> P: I have put five buttons on one chart and one on the other.
>
> T: Who else did it this way?
>
> The pupils raise their hands. The teacher calls one of them to the blackboard.
>
> T: How many buttons are on your first chart?
>
> P: Five buttons.
>
> T: Give me five circles.

The pupil gives him the circles.

The teacher sticks the circles on the blackboard, then asks: "How many buttons are on the other chart?"

P: One button.

T: Give me one circle.

The pupil gives him a circle.

The teacher places this circle on the blackboard, setting it apart from the other five circles, then asks: "How many buttons altogether do each of you have on your charts?"

P: Five buttons and one button, six buttons in all.

T: And how else can you place the six buttons? Who has placed them differently?

The children in turn describe their variants in placement of the buttons. Each example of the children's button placement is fixed on the blackboard.

We turn now to describing work on the mental composition of a number. This is accomplished with an aid consisting of small circles tied together on a string. The work is first conducted so that one of the summands is visually perceived by the children, the other must be found mentally.

T: I have six circles in my hands (he shows that he is holding the circles). There are four circles in one hand (opens his hand, showing the circles hanging on a string). How many circles are in the other hand? Who remembers how many circles I have altogether?

P: You have six circles.

T: How many in one hand?

P: Four circles in one hand.

T: How many circles in the other hand?

The pupils give the correct answer.

After all variants of the composition of the number being studied (say, 6) have been thus exhausted, we pass on to work on the composition of a number by imagining both component numbers. The same circles are used.

The teacher takes six circles in his hands and says: "Guess how many circles I took in one hand and how many in the other." The pupils give various suggestions. "And how can I take the circles in another way?" asks the teacher. When all variants of the composition of the number 6 have been suggested, he opens his hands and shows the children which of them "guessed" correctly.

When the children were acquainted with all cases of addition within the limits of the studied number, a table of addition was prepared. For best mastery, the teacher composed it together with the pupils at the lesson. The children did this with interest. They were convinced that they had already memorized many cases, but primarily that they knew how to explain each case of addition.

Subtraction was studied on the basis of addition. The teacher approached subtraction after the numbers from one thru five had been studied. This was done because the simultaneous study of addition and subtraction creates two difficulties for the children: they confuse these operations and their signs. Separate study of the operations eliminates these difficulties.

Study of subtraction, as of addition, was begun with an explanation of the sense of the operation and the mastery of the corresponding terms. As in addition, elucidation of the sense of subtraction was connected with a concrete situation and with small numbers.

All the work done in explaining the essence of addition was analogously repeated in subtraction. There was a small difference; the notation of the operation was introduced as soon as the teacher gave the children the concept of subtraction, since this helped distinguish subtraction and addition.

When the children had learned the essence of subtraction, they began to perform the operation.

The first case was the subtraction of units. In doing the operation, the children used a table with a row of numbers. They learned that in subtraction they go in the other direction along the numerical row, and that the numbers diminish. The latter fact is explained by a constant comparison of the minuend and the remainder.

Understanding the link between subtraction and addition was attained mainly through practice, in which the pupils, taking away a number, found the required answer on the table of addition. The teacher helped in many ways, explaining how the addition table should be used in subtraction, calling attention to the table each time.

Conclusion

As a result of experimental research we established several techniques for teaching arithmetic to mentally retarded children which affected the development of their thinking. These techniques are connected with the special problems in using visual aids at arithmetic lessons in the auxiliary school.

1. In order to form the ability to compare objects in mentally retarded children, one must utilize those same techniques of comparison which the normally developing child uses. The latter, approaching comparison, systemizes the objects being compared (and in counting objects he groups them, thus facilitating his work).

2. In teaching comparison by spatial features, the objects used first should be objects from the child's surrounding life. After this one may pass to the comparison of objects (from special aids) distinguished from each other by one spatial feature (the remaining features are identical). Then objects may be compared in which there are several different spatial features.

3. An important condition for successfully teaching the comparison of objects by one of their spatial features to mentally retarded school-children is the gradual advancement of the pupils from the comparison of objects sharply distinguished from each other in one feature to the comparison of objects that are more similar in the same feature. Such work on comparing objects by spatial features is concluded with establishment of the correlation the same as. An analogous technique of gradually reducing the difference between objects is used in comparing groups of objects and in comparing numbers. For initial comparison, groups of objects sharply distinguished by quantity are taken; then there is a transition to the comparison of quantities closer in number. The same with numbers: for initial comparison numbers sharply distinguished in magnitude are taken, then closer numbers. The work is concluded with establishment of the correlation as many as, when the difference between the numbers is zero.

4. In forming each new concept it is necessary to create in the children a broad base for generalization.

5. Mastery of computational techniques of numbers to ten is

facilitated to a significant extent by the use of special aids to ensure correct performance of the operations.

6. From the addition of two groups of objects one must pass to addition in which one group is given as directly perceived and the other as imagined. Then the children must be asked to do addition of quantities mentally,and finally, addition of abstract numbers.

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